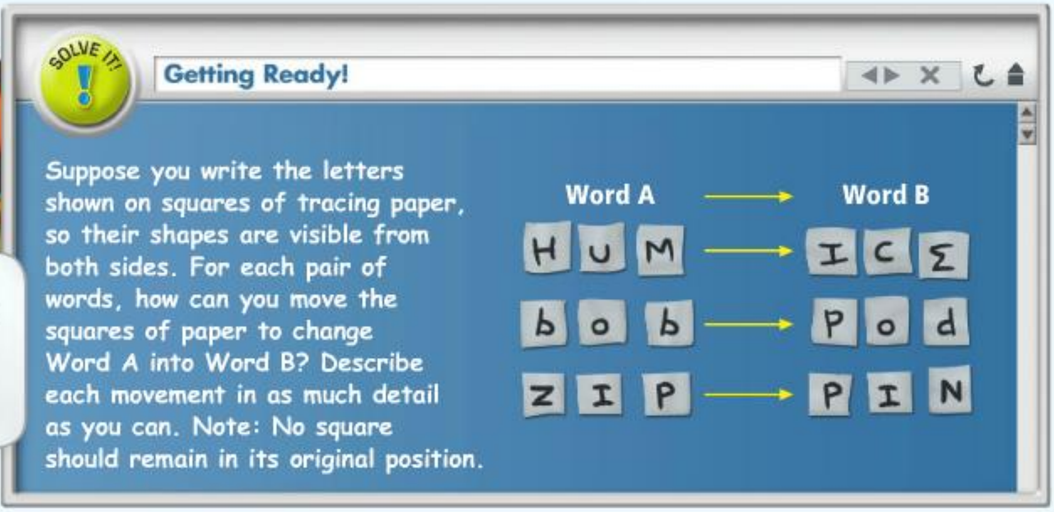
9-1 Translations

Objectives To identify isometries
To find translation images of figures

Think about how you can slide the pieces of paper around to turn Word A into Word B.



Suppose you write the letters shown on squares of tracing paper, so their shapes are visible from both sides. For each pair of words, how can you move the squares of paper to change Word A into Word B? Describe each movement in as much detail as you can. Note: No square should remain in its original position.

Lesson Vocabulary

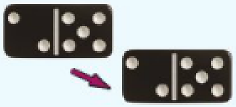
- transformation
- preimage
- image
- isometry
- translation
- composition of transformations

In the Solve It, you described changes in positions of letters. In this lesson, you will learn some of the mathematical language used to describe changes in positions of a geometric figure.

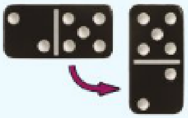
Essential Understanding You can change the position of a geometric figure without changing its size or shape.

A **transformation** of a geometric figure is a change in the position, shape, or size of the figure. When you play dominoes, you often move the dominoes by flipping them, sliding them, or turning them. Each move is a type of transformation. The diagrams below illustrate some basic transformations that you will study.

The domino flips.



The domino slides.



The domino turns.

In a transformation, the original figure is the **preimage.** The resulting figure is the **image.** An **isometry** is a transformation in which the preimage and image are congruent.

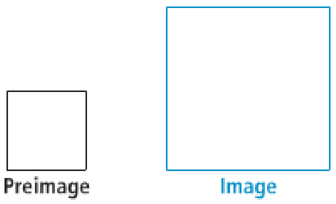
Think

What must be true about an isometry?
In an isometry, the image and the preimage must be congruent.

Problem 1 Identifying an Isometry

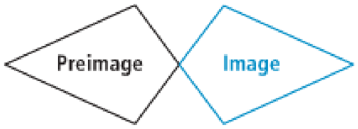
Does the transformation at the right appear to be an isometry? Explain.

No, this transformation involves a change in size. The sides of the preimage square and the sides of its image are not congruent.

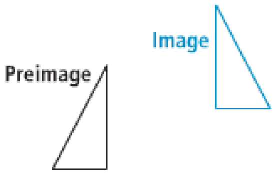


Got It? **1.** Does the transformation appear to be an isometry? Explain.

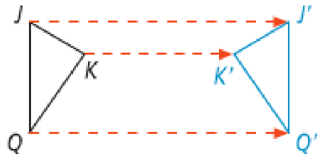
a.



b.



A transformation maps a figure onto its image and may be described with arrow notation (→). Prime notation (′) is sometimes used to identify image points. In the diagram below, K' is the image of K.



$\triangle JKQ \rightarrow \triangle J'K'Q'$
$\triangle JKQ$ maps onto $\triangle J'K'Q'$.

Notice that you list corresponding points of the preimage and image in the same order, as you do for corresponding points of congruent or similar figures.

Plan

How do you identify corresponding points?
You can use the statement $EFGH \rightarrow E'F'G'H'$. Corresponding points have the same position in the names of the preimage and image.

Problem 2 Naming Images and Corresponding Parts

In the diagram, $E'F'G'H'$ is an image of $EFGH$.

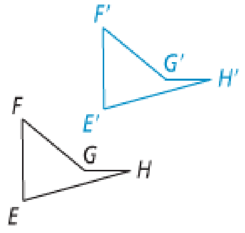
A What are the images of $\angle F$ and $\angle H$?

$\angle F'$ is the image of $\angle F$. $\angle H'$ is the image of $\angle H$.

B What are the pairs of corresponding sides?

$\overline{EF}$ and $\overline{E'F'}$ $\quad$ $\overline{FG}$ and $\overline{F'G'}$

$\overline{EH}$ and $\overline{E'H'}$ $\quad$ $\overline{GH}$ and $\overline{G'H'}$

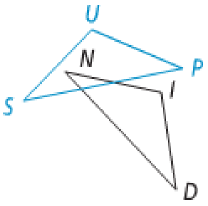


$EFGH \rightarrow E'F'G'H'$

Got It? **2.** In the diagram, $\triangle NID \rightarrow \triangle SUP$.

a. What are the images of $\angle I$ and point D?

b. What are the pairs of corresponding sides?



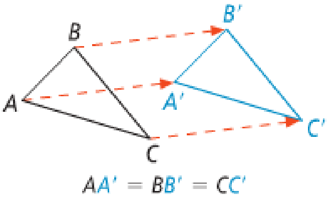
Essential Understanding The size, shape, and orientation of a geometric figure stay the same when you slide the figure in one direction.

take note

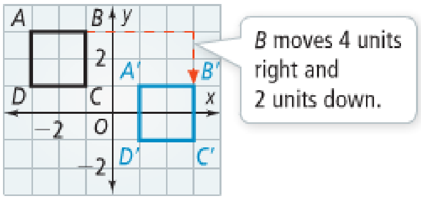
Key Concept Translation

A **translation** is a transformation that maps all points of a figure the same distance in the same direction.

A translation is an isometry.



The diagram at the right shows a translation in the coordinate plane. Each point of the black square moves 4 units right and 2 units down. Using variables, you can say that each (x, y) pair in the original figure is mapped to (x', y'), where $x' = x + 4$ and $y' = y - 2$. You can use arrow notation to write the following *translation rule*.

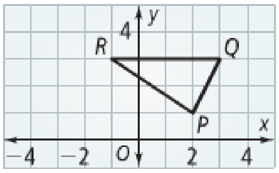


B moves 4 units right and 2 units down.

$$(x, y) \rightarrow (x + 4, y - 2)$$

Problem 3 Finding the Image of a Translation

What are the images of the vertices of $\triangle PQR$ for the translation $(x, y) \rightarrow (x - 2, y - 5)$? Graph the image of $\triangle PQR$.



Think

What does the rule tell you about the direction each point moves?

$x' = x - 2$ means that each point moves 2 units left. $y' = y - 5$ means that each point moves 5 units down.

Identify the coordinates of each vertex. Use the translation rule to find the coordinates of each vertex of the image.

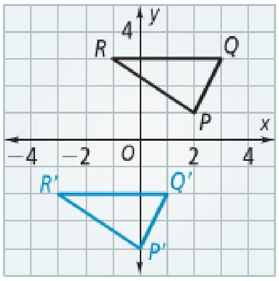
$(x, y) \rightarrow (x - 2, y - 5)$

$P(2, 1) \rightarrow (2 - 2, 1 - 5)$, or $P'(0, -4)$.

$Q(3, 3) \rightarrow (3 - 2, 3 - 5)$, or $Q'(1, -2)$.

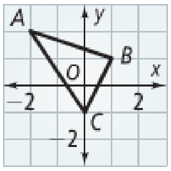
$R(-1, 3) \rightarrow (-1 - 2, 3 - 5)$, or $R'(-3, -2)$.

To graph the image of $\triangle PQR$, first graph P', Q', and R'. Then draw $\overline{P'Q'}$, $\overline{Q'R'}$, and $\overline{R'P'}$.



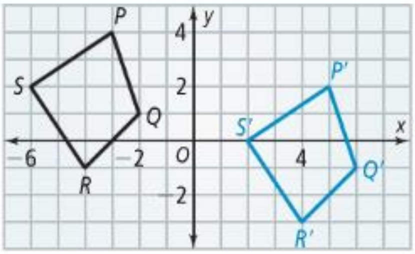
Got It? 3. a. What are the images of the vertices of $\triangle ABC$ for the translation $(x, y) \rightarrow (x + 1, y - 4)$? Graph $\triangle ABC$ and its image.

b. **Reasoning** Draw $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. What relationships exist among these three segments? How do you know?



Problem 4 Writing a Rule to Describe a Translation

What is a rule that describes the translation $PQRS \rightarrow P'Q'R'S'$?



Know

The coordinates of the vertices of both figures

Need

An algebraic relationship between (x, y) and (x', y')

Plan

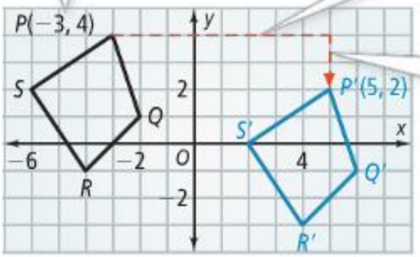
Use one pair of corresponding vertices to find the change in the horizontal direction (x) and the change in the vertical direction (y).

Think

How do you know which pair of corresponding vertices to use?

A translation moves all points the same distance and direction. You can use any pair of corresponding vertices.

Use $P(-3, 4)$ and its image $P'(5, 2)$.

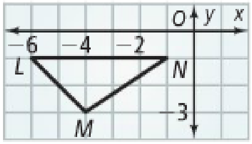


Horizontal change: $5 - (-3) = 8$
$x' = x + 8$

Vertical change: $2 - 4 = -2$
$y' = y - 2$

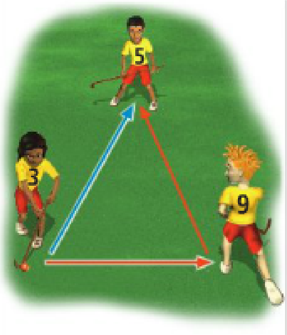
The translation rule is $(x, y) \rightarrow (x + 8, y - 2)$.

Got It? **4.** The translation image of $\triangle LMN$ is $\triangle L'M'N'$ with $L'(1, -2)$, $M'(3, -4)$, and $N'(6, -2)$. What is a rule that describes the translation?



A **composition of transformations** is a combination of two or more transformations. In a composition, you perform each transformation on the image of the preceding transformation.

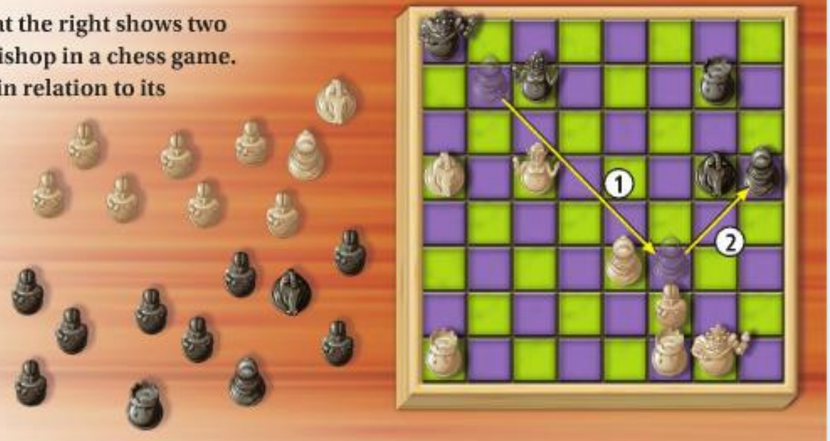
In the diagram at the right, the field hockey ball can move from Player 3 to Player 5 by a direct pass. This translation is represented by the blue arrow. The ball can also be passed from Player 3 to Player 9, and then from Player 9 to Player 5. The two red arrows represent this composition of translations.



In general, the composition of any two translations is another translation.

Problem 5 Composing Translations

Chess The diagram at the right shows two moves of the black bishop in a chess game. Where is the bishop in relation to its original position?



Use $(0, 0)$ to represent the bishop's original position. Write translation rules to represent each move.

$(x, y) \rightarrow (x + 4, y - 4)$ The bishop moves 4 squares right and 4 squares down.

$(x, y) \rightarrow (x + 2, y + 2)$ The bishop moves 2 squares right and 2 squares up.

The bishop's current position is the composition of the two translations.

First, $(0, 0)$ translates to $(0 + 4, 0 - 4)$, or $(4, -4)$.

Then, $(4, -4)$ translates to $(4 + 2, -4 + 2)$, or $(6, -2)$.

The bishop is 6 squares right and 2 squares down from its original position.

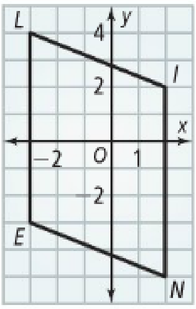
Got It? **5.** The bishop next moves 3 squares left and 3 squares down. Where is the bishop in relation to its original position?

Lesson Check

Do you know HOW?

1. If $\triangle JPT \rightarrow \triangle J'P'T'$, what is the image of P? The image of $\overline{TJ}$?

2. Copy the graph at the right. Graph the image of $NILE$ for the translation $(x, y) \rightarrow (x - 3, y - 4)$.

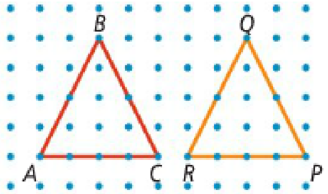


3. Point $H(x, y)$ moves 12 units left and 4 units up. What is a rule that describes this translation?

Do you UNDERSTAND?

4. **Vocabulary** What is true about a transformation that is not an isometry? Include a sketch of an example.

5. **Error Analysis** Your friend says the transformation $\triangle ABC \rightarrow \triangle PQR$ is a translation. Explain and correct her error.



6. **Reasoning** Write the translation $(x, y) \rightarrow (x + 1, y - 3)$ as a composition of a horizontal translation and a vertical translation.

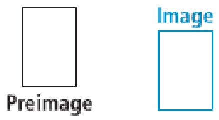
Practice and Problem-Solving Exercises

A Practice

Tell whether the transformation appears to be an isometry. Explain.

See Problem 1.

7.



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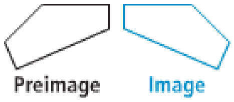

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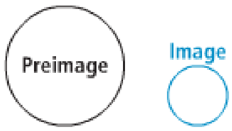
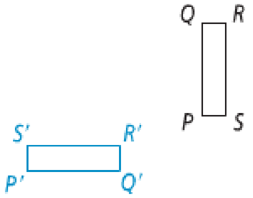


In each diagram, the blue figure is an image of the black figure.
(a) Choose an angle or point from the preimage and name its image.
(b) List all pairs of corresponding sides.

See Problem 2.

10.



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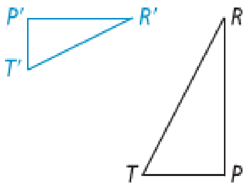

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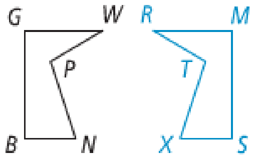
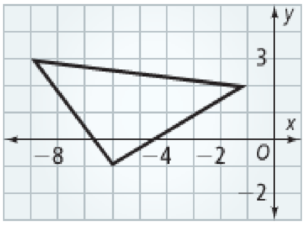


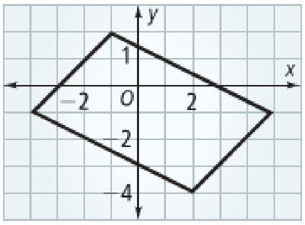
Copy each graph. Graph the image of each figure under the given translation.

See Problem 3.

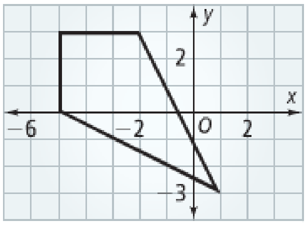
13. $(x, y) \rightarrow (x + 3, y + 2)$



14. $(x, y) \rightarrow (x + 5, y - 1)$



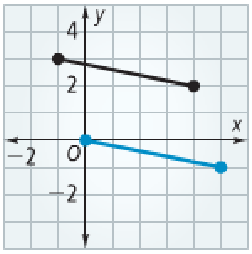
15. $(x, y) \rightarrow (x - 2, y + 5)$



The blue figure is a translation image of the black figure. Write a rule to describe each translation.

See Problem 4.

16.



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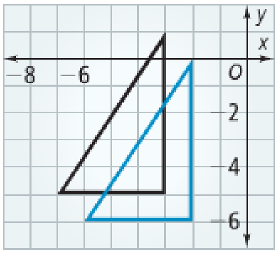

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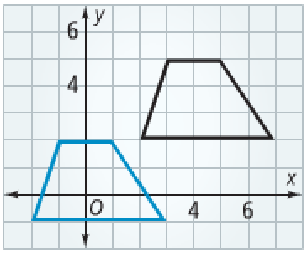


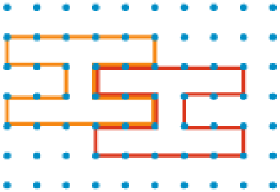
19. **Travel** You are visiting San Francisco. From your hotel near Union Square, you walk 4 blocks east and 4 blocks north to the Wells Fargo History Museum. Then you walk 5 blocks west and 3 blocks north to the Cable Car Barn Museum. Where is the Cable Car Barn Museum in relation to your hotel?

See Problem 5.

20. **Travel** Your friend and her parents are visiting colleges. They leave their home in Enid, Oklahoma, and drive to Tulsa, which is 107 mi east and 18 mi south of Enid. From Tulsa, they go to Norman, 83 mi west and 63 mi south of Tulsa. Where is Norman in relation to Enid?

B Apply

21. In the diagram at the right, the orange figure is a translation image of the red figure. Write a rule that describes the translation.



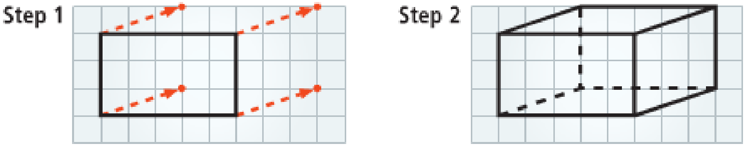
22. **Think About a Plan** $\triangle MUG$ has coordinates $M(2, -4)$, $U(6, 6)$, and $G(7, 2)$. A translation maps point M to $M'(-3, 6)$. What are the coordinates of U' and G' for this translation?

- How can you use a graph to help you visualize the problem?
- How can you find a rule that describes the translation?

23. **Coordinate Geometry** $PLAT$ has vertices $P(-2, 0)$, $L(-1, 1)$, $A(0, 1)$, and $T(-1, 0)$. $P'L'A'T'$ is the image of $PLAT$ for the translation $(x, y) \rightarrow (x + 2, y - 3)$. Show that $\overline{PP'}$, $\overline{LL'}$, $\overline{AA'}$, and $\overline{TT'}$ are all parallel.

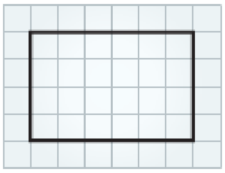
Geometry in 3 Dimensions **Follow the sample at the right. Use each figure, graph paper, and the given translation to draw a three-dimensional figure.**

SAMPLE Use the rectangle and $(x, y) \rightarrow (x + 3, y + 1)$ to draw a box.

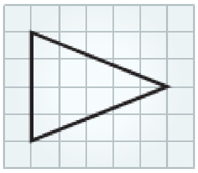


Step 1 Step 2

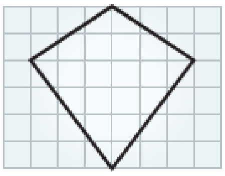
24. $(x, y) \rightarrow (x + 2, y - 1)$



25. $(x, y) \rightarrow (x - 2, y + 2)$



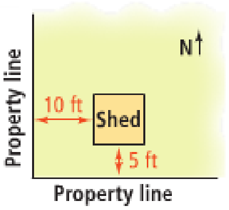
26. $(x, y) \rightarrow (x - 3, y - 5)$



27. **Open-Ended** You are a graphic designer for a company that manufactures wrapping paper. Make a design for wrapping paper that involves translations.

28. **Reasoning** The translation $(x, y) \rightarrow (x + 5, y + 7)$ maps $\triangle MNO$ onto $\triangle M'N'O'$. What translation rule maps $\triangle M'N'O'$ onto $\triangle MNO$?

29. **Landscaping** The diagram at the right shows the site plan for a backyard storage shed. Local law, however, requires the shed to sit at least 15 ft from property lines. Describe how to move the shed to comply with the law.

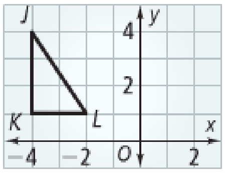


30. **Computer Animation** You write a computer animation program to help young children learn the alphabet. The program draws a letter, erases the letter, and makes it reappear in a new location two times. The program uses the following composition of translations to move the letter.

$(x, y) \rightarrow (x + 5, y + 7)$ followed by $(x, y) \rightarrow (x - 9, y - 2)$

Suppose the program makes the letter W by connecting the points (1, 2), (2, 0), (3, 2), (4, 0) and (5, 2). What points does the program connect to make the last W?

31. Use the graph at the right. Write three different translation rules for which the image of $\triangle JKL$ has a vertex at the origin.



Find a translation that has the same effect as each composition of translations.

32. $(x, y) \rightarrow (x + 2, y + 5)$ followed by $(x, y) \rightarrow (x - 4, y + 9)$

33. $(x, y) \rightarrow (x + 12, y + 0.5)$ followed by $(x, y) \rightarrow (x + 1, y - 3)$

Challenge

34. Coordinate Geometry $\triangle ABC$ has vertices $A(-2, 5)$, $B(-4, -1)$, and $C(2, -3)$. Show that the images of the midpoints of the sides of $\triangle ABC$ are the midpoints of the sides of $\triangle A'B'C'$ for the translation $(x, y) \rightarrow (x + 4, y + 2)$.

35. Writing Explain how to use translations to draw a parallelogram.

Standardized Test Prep

SAT/ACT

36. $\triangle ABC$ has vertices $A(-5, 2)$, $B(0, -4)$, and $C(3, 3)$. What are the vertices of the image of $\triangle ABC$ under the translation $(x, y) \rightarrow (x + 7, y - 5)$?

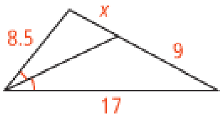
Ⓐ $A'(2, -3)$, $B'(7, -9)$, $C'(10, -2)$

Ⓑ $A'(-12, -3)$, $B'(-7, -9)$, $C'(-4, -2)$

Ⓒ $A'(-12, 7)$, $B'(-7, 1)$, $C'(-4, 8)$

Ⓓ $A'(2, -3)$, $B'(10, -2)$, $C'(7, -9)$

37. What is the value of x in the figure at the right?



Ⓕ 4.5

Ⓖ 16

Ⓗ 18

Ⓘ 18.5

38. In $\triangle PQR$, $PQ = 4.5$, $QR = 4.4$, and $RP = 4.6$. Which statement is true?

Ⓐ $m\angle P + m\angle Q < m\angle R$

Ⓑ $\angle Q$ is the largest angle.

Ⓒ $\angle R$ is the largest angle.

Ⓓ $m\angle R < m\angle P$

Short Response

39. ▱$ABCD$ has vertices $A(0, -3)$, $B(-4, -2)$, and $D(-1, 1)$.

a. What are the coordinates of C?

b. Is ▱$ABCD$ a rhombus? Explain.

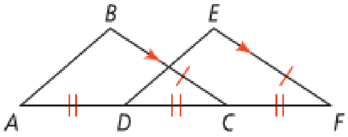
Mixed Review

40. Navigation An airplane lands at a point 100 km east and 420 km south from where it took off. Describe the magnitude and the direction of its flight vector.

See Lesson 8-5.

41. Given: $\overline{BC} \cong \overline{EF}$, $\overline{BC} \parallel \overline{EF}$,
$\overline{AD} \cong \overline{DC} \cong \overline{CF}$

Prove: $\overline{AB} \cong \overline{DE}$



See Lesson 4-7.

Get Ready! **To prepare for Lesson 9-2, do Exercises 42–44.**

Write an equation for the line through A perpendicular to the given line.

See Lesson 3-8.

42. $A(1, -2)$; $x = -2$

43. $A(-1, -1)$; $y = 1$

44. $A(-1, 2)$; $y = x$

Concept Byte

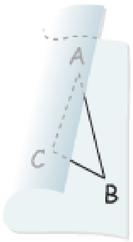
Use With Lesson 9-2

ACTIVITY

Paper Folding and Reflections

In Activity 1, you will see how a figure and its *reflection* image are related. In Activity 2, you will use these relationships to construct a reflection image.

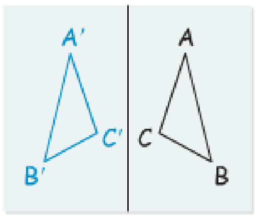
Activity 1

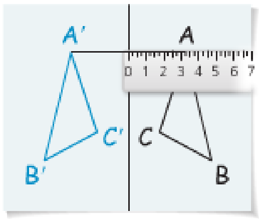


Step 1 Use a piece of tracing paper and a straightedge. Using less than half the page, draw a large, scalene triangle. Label its vertices A, B, and C.

Step 2 Fold the paper so that your triangle is covered. Trace $\triangle ABC$ using a straightedge.

Step 3 Unfold the paper. Label the traced points corresponding to A, B, and C as A', B', and C', respectively. $\triangle A'B'C'$ is a reflection image of $\triangle ABC$. The fold is the reflection line.





1. Use a ruler to draw $\overline{AA'}$. Measure the perpendicular distances from A to the fold and from A' to the fold. What do you notice?

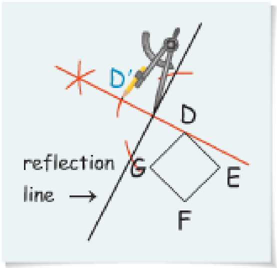
2. Measure the angles formed by the fold and $\overline{AA'}$.

3. Repeat Questions 1 and 2 for B and B' and for C and C'. Then, make a conjecture: How is the reflection line related to the segment joining a point and its image?

Activity 2

Step 1 On regular paper, draw a simple shape or design made of segments. Use less than half the page. Draw a reflection line near your figure.

Step 2 Use a compass and straightedge to construct a perpendicular to the reflection line through one point of your drawing.



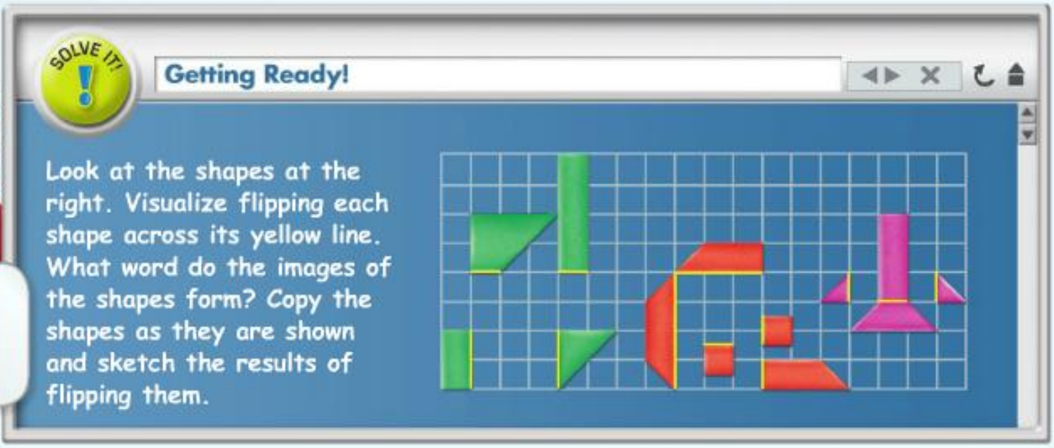
4. Explain how you can use a compass and the perpendicular you drew to find the reflection image of the point you chose.

5. Connect the reflection images for several points of your shape and complete the image. Check the accuracy of the reflection image by folding the paper along the reflection line and holding it up to a light source.

9-2 Reflections

Objective To find reflection images of figures

Flipping a figure across a line is another type of transformation.



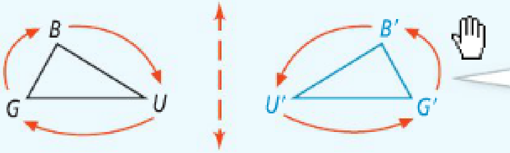
Look at the shapes at the right. Visualize flipping each shape across its yellow line. What word do the images of the shapes form? Copy the shapes as they are shown and sketch the results of flipping them.

Dynamic Activity
Reflections

Lesson Vocabulary
- reflection
- line of reflection

In the Solve It, you visualized shapes flipping across lines. Notice that when a figure flips across a line, the preimage and its image are congruent and have *opposite orientations*. You can tell that two figures have opposite orientations if the corresponding vertices of the preimage and image read in opposite directions.

The vertices of $\triangle BUG$ read clockwise.



The vertices of $\triangle B'U'G'$ read counterclockwise.

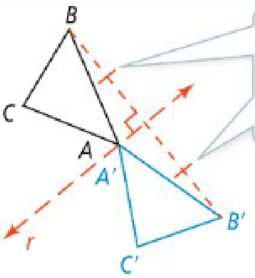
Essential Understanding The size and shape of a geometric figure stay the same when you flip the figure across a line. Its orientation reverses.

take note Key Concept Reflection Across a Line

Reflection across a line r, called the **line of reflection,** is a transformation with these two properties:

- If a point A is on line r, then the image of A is itself (that is, $A' = A$).
- If a point B is not on line r, then r is the perpendicular bisector of $\overline{BB'}$.

A reflection across a line is an isometry.



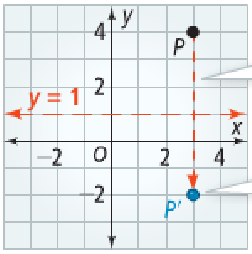
The preimage B and its image B' are equidistant from the line of reflection.

Problem 1 Reflecting a Point Across a Line

Multiple Choice **If point $P(3, 4)$ is reflected across the line $y = 1$, what are the coordinates of its reflection image?**

Ⓐ $(3, -4)$ Ⓑ $(0, 4)$ Ⓒ $(3, -2)$ Ⓓ $(-3, -2)$

Graph point P and the line of reflection $y = 1$. P and its reflection image across the line must be equidistant from the line.



Move along the line through P that is perpendicular to the line of reflection.

Stop when the distances of P and P' to the line of reflection are the same.

Think

How does a graph help you visualize the problem?

A graph shows that $y = 1$ is a horizontal line, so the line through P that is perpendicular to the line of reflection is a vertical line.

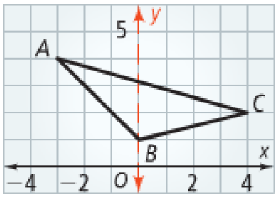
P is 3 units above the line $y = 1$, so P' is 3 units below the line $y = 1$. The line $y = 1$ is the perpendicular bisector of $\overline{PP'}$ if P' is $(3, -2)$. The correct answer is C.

Got It? **1.** What is the image of $P(3, 4)$ reflected across the line $x = -1$?

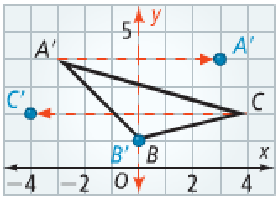
Problem 2 Graphing a Reflection Image

Coordinate Geometry **Graph points $A(-3, 4)$, $B(0, 1)$, and $C(4, 2)$. What is the image of $\triangle ABC$ reflected across the y-axis?**

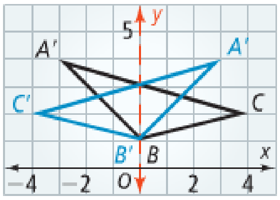
Step 1
Graph $\triangle ABC$. Show the y-axis as the dashed line of reflection.



Step 2
Find A', B', and C'. B' is in the same position as B because B is on the line of reflection. Locate A' and C' so that the y-axis is the perpendicular bisector of $\overline{AA'}$ and $\overline{CC'}$.



Step 3
Draw $\triangle A'B'C'$.



Think

$\triangle ABC$ intersects the line of reflection. How will the image relate to the line of reflection?

The image will also intersect the line of reflection.

Got It? **2.** Graph $\triangle ABC$ from Problem 2. What is the image of $\triangle ABC$ reflected across the x-axis?

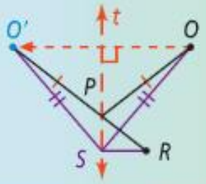
Problem 3 Minimizing a Distance

Hiking **Beginning from a point on Summit Trail, a hiking club will build a trail to the Overlook and a trail to Balance Rock. Working under a tight budget, the club members want to minimize the total length of the two trails. How can you find the point on Summit Trail where the two new trails should start? Assume the trails will be straight and that they will cover similar terrain.**

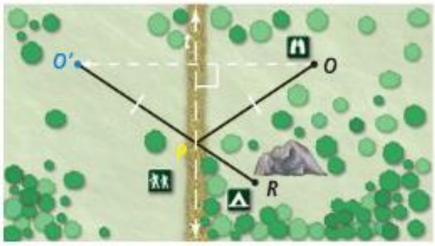
Know	Need	Plan
The ending points, *O* and *R*, of the two trails	The point *P* on Summit Trail for which *OP* + *PR* is as small as possible	Reflect *O* across Summit Trail to *O′*. Then find the intersection of Summit Trail and $\overline{RO'}$.

Think

To see how the Triangle Inequality Theorem applies, look at another possible starting point, *S*. You can use △*SRO′* to explain why *O′S* + *SR* > *O′P* + *PR*.



You need to find the point P on line t such that $OP + PR$ is as small as possible. Locate O', the reflection image of O across t. Because t is the perpendicular bisector of $\overline{OO'}$, $PO = PO'$ and $OP + PR = O'P + PR$. By the Triangle Inequality Theorem, the sum $O'P + PR$ is smallest when R, P, and O' are collinear. So, the trails should start at the point P where $\overline{RO'}$ intersects line t.



Got It? **3.** **Reasoning** Your classmate began to solve the problem above by reflecting point R in line t. Will her method work? Explain.

Lesson Check

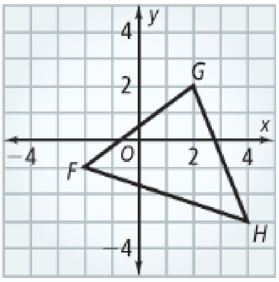
Do you know HOW?

Use the graph of $\triangle FGH$.

1. What are the coordinates of H reflected across the y-axis?

2. What are the coordinates of G reflected across the line $x = 3$?

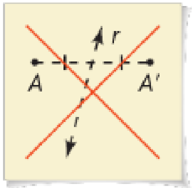
3. Graph $\triangle FGH$ and its reflection image across the line $y = 4$.



Do you UNDERSTAND?

4. **Vocabulary** What is the relationship between a line of reflection and a segment joining corresponding points of the preimage and image?

5. **Error Analysis** A classmate reflected point A across line r as shown in the diagram. Explain your classmate's error. Copy point A and line r and show the correct location of A'.



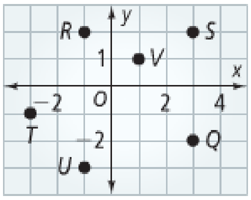
6. What are the coordinates of a point $P(x, y)$ reflected across the y-axis? Across the x-axis?

Practice and Problem-Solving Exercises

A Practice

Each point is reflected across the line indicated. Find the coordinates of each image.

See Problem 1.



7. Q across $x = 1$

8. R across $y = -1$

9. S across the y-axis

10. T across $y = 0.5$

11. U across $x = -3$

12. V across the x-axis

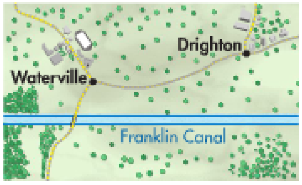
Coordinate Geometry **Given points $J(1, 4)$, $A(3, 5)$, and $R(2, 1)$, graph $\triangle JAR$ and its reflection image across each line.**

See Problem 2.

13. the x-axis **14.** the y-axis **15.** $y = 2$ **16.** $y = 5$ **17.** $x = -1$ **18.** $x = 2$

19. Civil Engineering Town officials in Waterville and Drighton plan to construct a water pumping station along the Franklin Canal. The station will provide both towns with water. Where along the canal should the officials build the pumping station to minimize the total length of pipe needed?

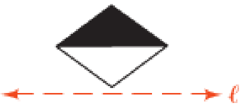
See Problem 3.



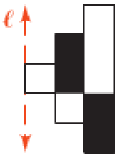
B Apply

Copy each figure and line ℓ. Draw each figure's reflection image across line ℓ.

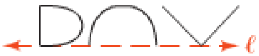
20.



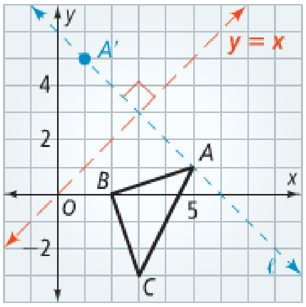
21.



22.



23. Coordinate Geometry The following steps explain how to reflect point A across the line $y = x$.



Step 1 Draw line ℓ through $A(5, 1)$ perpendicular to the line $y = x$. The slope of $y = x$ is 1, so the slope of line ℓ is $1 \cdot (-1)$, or -1.

Step 2 From A, move two units left and two units up to $y = x$. Then move two more units left and two more units up to find the location of A' on line ℓ. The coordinates of A' are $(1, 5)$.

a. Copy the diagram. Then draw the lines through B and C that are perpendicular to the line $y = x$. What is the slope of each line?

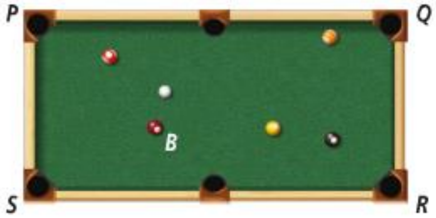
b. Reflect B and C across the line $y = x$. What are the coordinates of B' and C'?

c. Graph $\triangle A'B'C'$.

d. Make a Conjecture Compare the coordinates of the vertices of $\triangle ABC$ and $\triangle A'B'C'$. Make a conjecture about the coordinates of the point $P(a, b)$ reflected across the line $y = x$.

24. Coordinate Geometry $\triangle ABC$ has vertices $A(-3, 5)$, $B(-2, -1)$, and $C(0, 3)$. Graph $\triangle ABC$ and $\triangle A'B'C'$, the reflection of $\triangle ABC$ across the line $y = -x$. (*Hint:* See Exercise 23.)

25. **Recreation** When you play pool, you can use the fact that the ball bounces off the side of the pool table at the same angle at which it hits the side. Suppose you want to put the ball at point B into the pocket at point P by bouncing it off side $\overline{RS}$. Off what point on $\overline{RS}$ should the ball bounce? Draw a diagram and explain your reasoning.

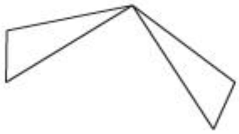


26. **Think About a Plan** The coordinates of the vertices of $\triangle FGH$ are $F(2, -1)$, $G(-2, -2)$, and $H(-4, 3)$. Graph $\triangle FGH$ and its image $\triangle F'G'H'$ for a reflection across the line $y = x - 3$.

- What is the relationship between the line $y = x - 3$ and $\overline{FF'}$, $\overline{GG'}$, and $\overline{HH'}$?
- How can you use slope to find the image of each vertex?

Copy each pair of figures. Then draw the line of reflection you can use to map one figure onto the other.

27.



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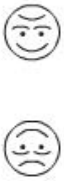

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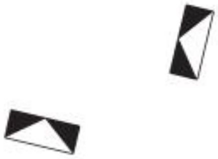
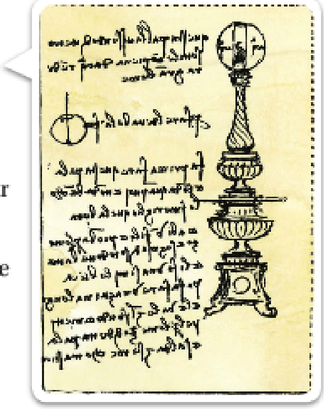


30. **History** The work of artist and scientist Leonardo da Vinci (1452–1519) has an unusual characteristic. His handwriting is a mirror image of normal handwriting.

a. Write the mirror image of the sentence, “Leonardo da Vinci was left-handed.” Use a mirror to check how well you did.

b. Explain why the fact about da Vinci in part (a) might have made mirror writing seem natural to him.



31. **Open-Ended** Give three examples from everyday life of objects that come in a left-handed version and a right-handed version.

Find the image of $O(0, 0)$ after two reflections, first across line ℓ_1 and then across line ℓ_2.

32. $\ell_1: y = 3$, ℓ_2: x-axis

33. $\ell_1: x = -2$, ℓ_2: y-axis

34. ℓ_1: x-axis, ℓ_2: y-axis

35. $\ell_1: x = -2, \ell_2: y = 3$

36. $\ell_1: y = 3, \ell_2: x = -2$

37. $\ell_1: x = -2, \ell_2: y = x$

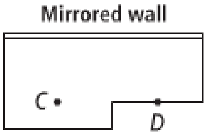
38. $\ell_1: x = a, \ell_2: y = b$

39. $\ell_1: x = a, \ell_2: y = x$

40. $\ell_1: y = b, \ell_2: y = x$

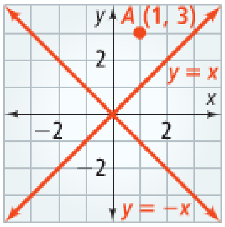
41. **Reasoning** When you reflect a figure across a line, does every point on the preimage move the same distance? Explain.

42. **Security** Recall that when a ray of light hits a mirror, it bounces off the mirror at the same angle at which it hits the mirror. You are installing a security camera. At what point on the mirrored wall should you aim the camera at C in order to view the door at D? Draw a diagram and explain your reasoning.



43. Use the diagram at the right. Find the coordinates of the given point.

- **a.** A', the reflection image of A across $y = x$
- **b.** A'', the reflection image of A' across $y = -x$
- **c.** A''', the reflection image of A'' across $y = x$
- **d.** A'''', the reflection image of A''' across $y = -x$
- **e.** How are A and A'''' related?



Challenge

Reasoning **Can you form the given type of quadrilateral by drawing a triangle and then reflecting one or more times? Explain.**

44. parallelogram

45. isosceles trapezoid

46. kite

47. rhombus

48. rectangle

49. square

50. **Coordinate Geometry** Show that $B(b, a)$ is the reflection image of $A(a, b)$ across the line $y = x$. (*Hint:* Show that $y = x$ is the perpendicular bisector of $\overline{AB}$.)

Standardized Test Prep

SAT/ACT

51. What is the reflection image of (a, b) across the line $y = -6$?

Ⓐ $(a - 6, b)$ Ⓑ $(a, b - 6)$ Ⓒ $(-12 - a, b)$ Ⓓ $(a, -12 - b)$

52. The diagonals of a quadrilateral are perpendicular and bisect each other. What is the most precise name for the quadrilateral?

Ⓕ rectangle Ⓖ parallelogram Ⓗ rhombus Ⓘ kite

53. $\overrightarrow{AD}$ bisects $\angle A$ of $\triangle CAB$, with point D on $\overline{CB}$. Which of the following is true?

Ⓐ $\frac{CD}{DB} = \frac{CA}{BA}$ Ⓑ $\frac{CD}{DB} = \frac{BA}{CA}$ Ⓒ $\frac{CB}{CD} = \frac{CA}{BA}$ Ⓓ $\frac{CA}{CB} = \frac{AB}{DB}$

Extended Response

54. Write an indirect proof of the following statement: The hypotenuse of a right triangle is the longest side of the right triangle.

Mixed Review

For the given points, $\overline{A'B'}$ is a translation of $\overline{AB}$. Write a rule to describe each translation. **See Lesson 9-1.**

55. $A(-1, 5)$, $B(2, 0)$, $A'(3, 3)$, $B'(6, -2)$

56. $A(-9, -4)$, $B(-7, 1)$, $A'(-4, -3)$, $B'(-2, 2)$

57. **Maps** A map of Alberta, Canada, uses the scale 1 cm = 25 km. On the map, the distance from Calgary to Edmonton is 11.1 cm. How far apart are the two cities? **See Lesson 7-2.**

Get Ready! **To prepare for Lesson 9-3, do Exercises 58–63.**

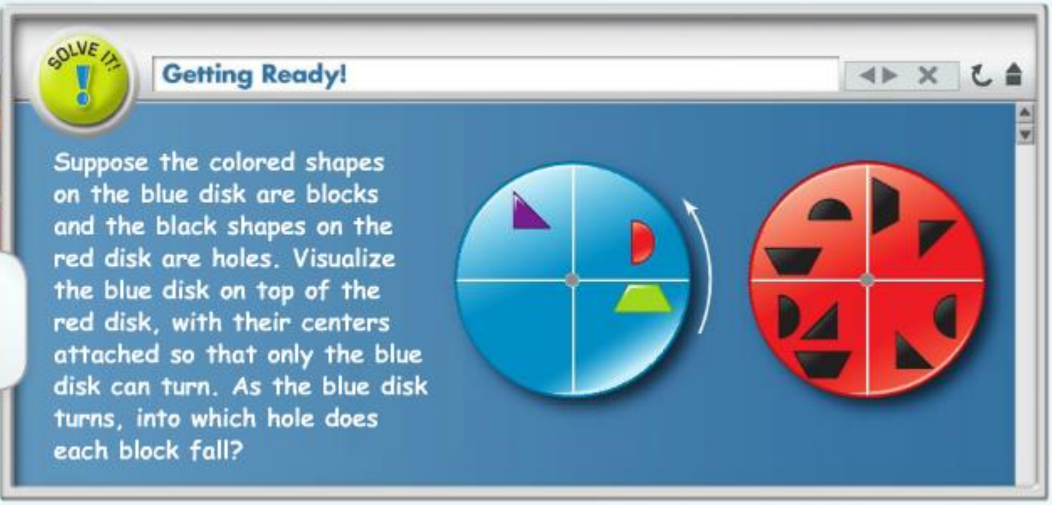
Use a protractor to draw an angle with the given measure. **See p. 824.**

58. 120° **59.** 90° **60.** 72° **61.** 60° **62.** 45° **63.** 36°

9-3 Rotations

Objective To draw and identify rotation images of figures

Keep your eye on one corner of a shape and follow its path.



Getting Ready!

Suppose the colored shapes on the blue disk are blocks and the black shapes on the red disk are holes. Visualize the blue disk on top of the red disk, with their centers attached so that only the blue disk can turn. As the blue disk turns, into which hole does each block fall?

Dynamic Activity
Rotations, Reflections, and Translations

Lesson Vocabulary
- rotation
- center of rotation
- angle of rotation
- center of a regular polygon

In the Solve It, you thought about how the shapes move as the disk turns, or *rotates*, about its center point. In this lesson, you will learn how to recognize and construct rotations of geometric figures.

Essential Understanding The size, shape, and orientation of a geometric figure stays the same when you turn the figure about a point.

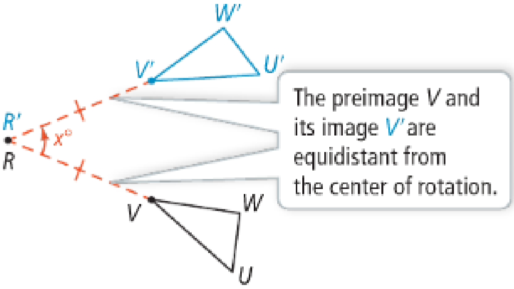
take note Key Concept Rotation About a Point

A **rotation** of $x°$ about a point R, called the **center of rotation,** is a transformation with these two properties:

- The image of R is itself (that is, $R' = R$).
- For any other point V, $RV' = RV$ and $m\angle VRV' = x$.

The positive number of degrees a figure rotates is the **angle of rotation.**

A rotation about a point is an isometry.

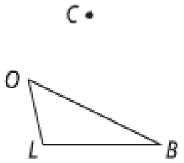


The preimage V and its image V' are equidistant from the center of rotation.

Unless stated otherwise, rotations in this book are counterclockwise.

Problem 1 Drawing a Rotation Image

What is the image of $\triangle LOB$ for a 100° rotation about C?



Plan

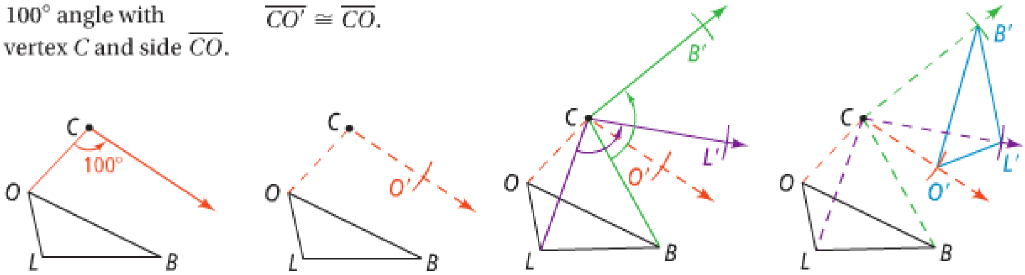
How do you use the definition of rotation about a point to help you get started?
You know that *O* and *O′* must be equidistant from *C* and that $m\angle OCO'$ must be 100.

Step 1
Draw $\overline{CO}$. Use a protractor to draw a 100° angle with vertex C and side $\overline{CO}$.

Step 2
Use a compass to construct $\overline{CO'} \cong \overline{CO}$.

Step 3
Locate B' and L' in a similar manner.

Step 4
Draw $\triangle L'O'B'$.

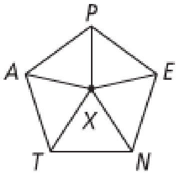


Got It? **1.** Copy $\triangle LOB$ from Problem 1. What is the image of $\triangle LOB$ for a 50° rotation about B?

The **center of a regular polygon** is the point that is equidistant from its vertices. The center and the vertices of a regular n-gon determine n congruent triangles. You can use this fact to find rotation images of regular polygons.

Problem 2 Identifying a Rotation Image

Point X is the center of regular pentagon $PENTA$. What is the image of the given point or segment for the given rotation?

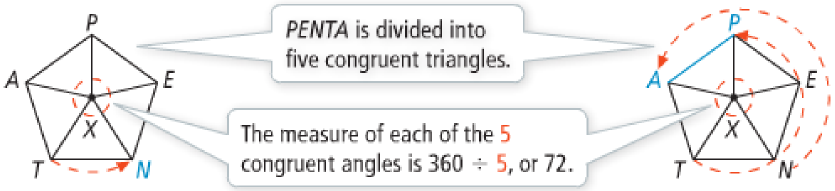


Think

Why are the five triangles congruent?
The center of a regular polygon is equidistant from the vertices, so the five segments from *X* are congruent. The fives sides of *PENTA* are congruent because *PENTA* is regular. So the triangles are all congruent by SSS.

A **72° rotation of T about X**

B **216° rotation of $\overline{TN}$ about X**



PENTA is divided into five congruent triangles.

The measure of each of the 5 congruent angles is 360 ÷ 5, or 72.

When T rotates 72° about X, it moves one vertex counterclockwise. The image of T is N.

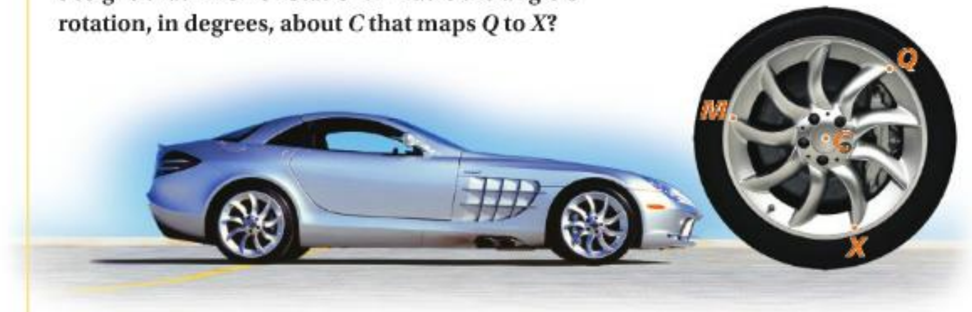
$216 \div 72 = 3$, so a 216° rotation about X moves each vertex three vertices counterclockwise. The image of $\overline{TN}$ is $\overline{PA}$.

Got It? **2.** What is the image of E for a 144° rotation about X?

Problem 3 Finding an Angle of Rotation

GRIDDED RESPONSE

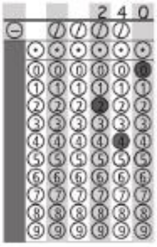
Design **Hubcaps of car wheels often have interesting designs that involve rotation. What is the angle of rotation, in degrees, about *C* that maps *Q* to *X*?**



Think

Can you visualize the nine 40° angles at the center?
Each angle would be formed by joining the tips of two consecutive spokes to *C*.

The design consists of 9 spokes, so it divides the circle into 9 congruent parts. Since $360 \div 9 = 40$, each part has a 40° angle at the center. Moving counterclockwise, *Q* touches 6 spokes as it rotates to point *X*. The angle of rotation that maps *Q* to *X* is $6 \cdot 40°$, or 240°.



Got It? **3. a.** In Problem 3, what is the angle of rotation about *C* that maps *M* to *Q*?

b. **Reasoning** If a 50° rotation about a point *P* maps point *A* to point *B*, what angle of rotation maps point *B* to point *A*?

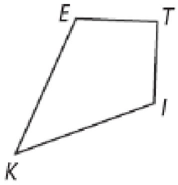
A composition of rotations about the same point is itself a rotation about that point. To sketch the image, add the angles of rotation to find the total rotation.

Problem 4 Finding a Composition of Rotations

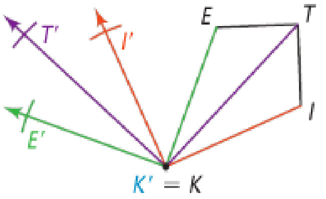
Plan

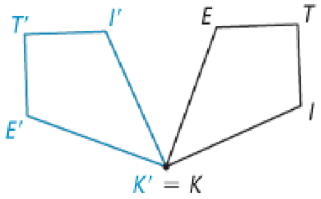
Do you need to do two rotations?
No. Both rotations are about the same point, so the angle of rotation of the composition is equal to the sum of the two angles of rotation.

What is the image of *KITE* for a composition of a 30° rotation and a 60° rotation, both about point *K*?



The angle of rotation for the composition is $30° + 60° = 90°$. Draw *KITE*. Locate image points of the vertices for a 90° rotation. Use the image points to sketch the entire image.



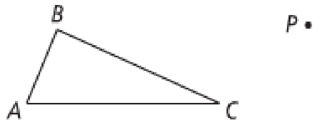


Got It? **4.** What are the coordinates of the image of point $A(-2, 3)$ for a composition of two 90° rotations about the origin?

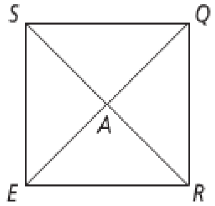
Lesson Check

Do you know HOW?

1. Copy the figure and point P. Draw the image of $\triangle ABC$ rotated 70° about P.



In the figure below, point A is the center of square $SQRE$.



2. What is the image of E for a 90° rotation about A?

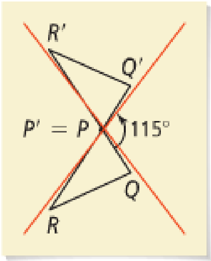
3. What is the image of $\overline{RQ}$ for a 180° rotation about A?

4. What is the image of S for a composition of a 30° rotation and a 240° rotation, both about point A?

Do you UNDERSTAND?

5. **Vocabulary** $\triangle A'B'C'$ is a rotation image of $\triangle ABC$ about point O. Describe how to find the angle of rotation.

6. **Error Analysis** A classmate drew a 115° rotation of $\triangle PQR$ about point P, as shown at the right. Explain and correct your classmate's error.



7. **Compare and Contrast** Compare rotating a figure about a point to reflecting the figure across a line. How are the transformations alike? How are they different?

8. **Reasoning** Point $P(x, y)$ is rotated about the origin by 135° and then by 45°. What are the coordinates of the image of point P?

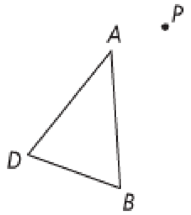
Practice and Problem-Solving Exercises

A Practice

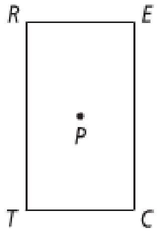
Copy each figure and point P. Draw the image of each figure for the given rotation about P. Use prime notation to label the vertices of the image.

See Problem 1.

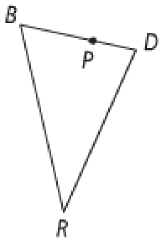
9. 60°



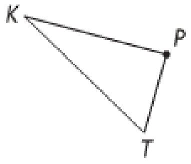
10. 90°



11. 180°

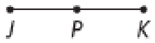


12. 90°

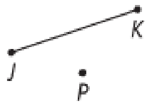


Copy each figure and point P. Then draw the image of $\overline{JK}$ for a 180° rotation about P. Use prime notation to label the vertices of the image.

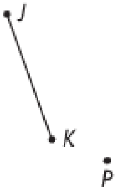
13.



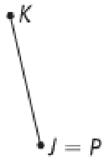
14.



15.



16.



Point O is the center of regular hexagon $HEXAGN$. Find the image of the given point or segment for the given rotation.

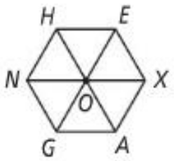
See Problem 2.

17. 60° rotation of E about O

18. 120° rotation of $\overline{AX}$ about O

19. 240° rotation of $\overline{NG}$ about O

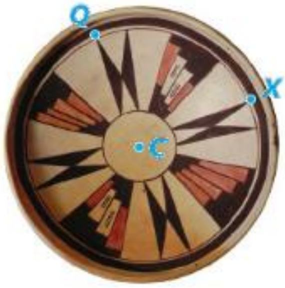
20. 360° rotation of H about O



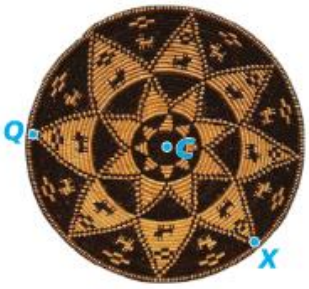
Native American Art **Find the angle of rotation about C that (a) maps Q to X and (b) maps X to Q.**

See Problem 3.

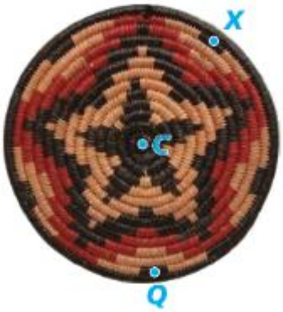
21.



22.



23.



For Exercises 24–29, copy $\triangle XYZ$. Draw the image of $\triangle XYZ$ for the given composition of rotations about the given point.

See Problem 4.

24. 45°, then 45°; X

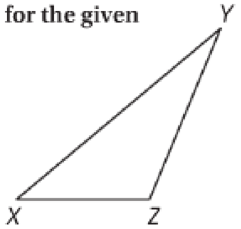
25. 45°, then 45°; Y

26. 30°, then 30°; Z

27. 20°, then 160°; Z

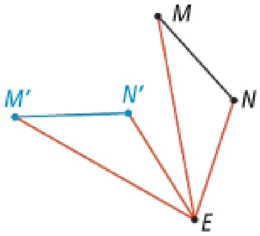
28. 135°, then 135°; Y

29. 180°, then 180°; X



B Apply

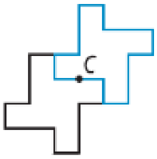
30. In the diagram at the right, $\overline{M'N'}$ is the rotation image of $\overline{MN}$ about point E. Name all pairs of congruent angles and all pairs of congruent segments in the diagram.



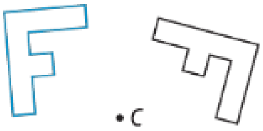
31. **Language Arts** The symbol ə is called a *schwa*. It is used in dictionaries to represent neutral vowel sounds such as *a* in *ago*, *i* in *sanity*, and *u* in *focus*. What transformation maps a ə to a lowercase e?

Find the angle of rotation about C that maps the black figure to the blue figure.

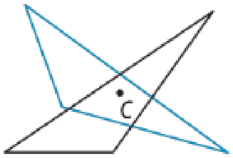
32.



33.

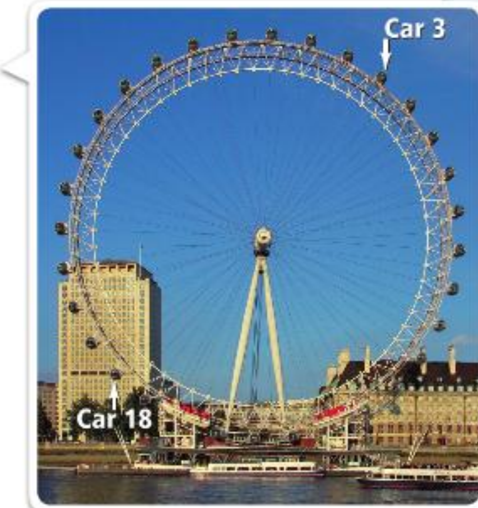


34.



35. **Think About a Plan** The Millenium Wheel, also known as the London Eye, contains 32 observation cars. Determine the angle of rotation that will bring Car 3 to the position of Car 18.

- How do you find the angle of rotation that a car travels when it moves one position counterclockwise?
- How many positions does Car 3 move?

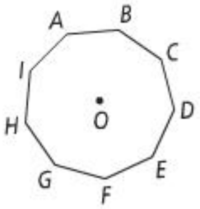


36. **Reasoning** For center of rotation P, does an $x°$ rotation followed by a $y°$ rotation give the same image as a $y°$ rotation followed by an $x°$ rotation? Explain.

37. **Writing** Describe compositions of rotations that have the same effect as a 360° rotation about a point X.

38. **Coordinate Geometry** Graph $A(5, 2)$. Graph B, the image of A for a 90° rotation about the origin O. Graph C, the image of A for a 180° rotation about O. Graph D, the image of A for a 270° rotation about O. What type of quadrilateral is $ABCD$? Explain.

Point O is the center of the regular nonagon shown at the right.

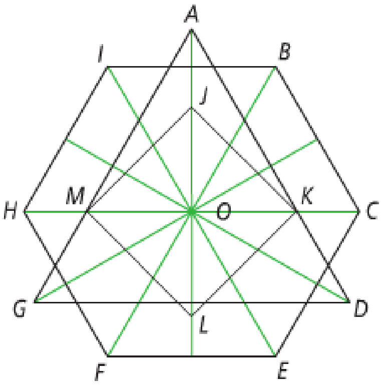


39. Find the angle of rotation that maps F to H.

40. **Open-Ended** Describe a composition of rotations that maps H to C.

41. **Error Analysis** Your friend says that $\overline{AB}$ is the image of $\overline{ED}$ for a 120° rotation about O. What is wrong with your friend's statement?

In the figure at the right, the large triangle, the quadrilateral, and the hexagon are regular. Find the image of each point or segment for the given rotation or composition of rotations. (*Hint:* Adjacent green segments form 30° angles.)



42. 120° rotation of B about O

43. 270° rotation of L about O

44. 300° rotation of $\overline{IB}$ about O

45. 60° rotation of E about O

46. 180° rotation of $\overline{JK}$ about O

47. 240° rotation of G about O

48. 120° rotation of F about H

49. 270° rotation of M about L

50. 60° rotation followed by 120° rotation of I about O

51. 45° rotation followed by 225° rotation of M about O

Challenge

52. **Reasoning** If you are given a figure and a rotation image of the figure, how can you find the center and angle of rotation?

53. **Coordinate Geometry** Draw $\triangle LMN$ with vertices $L(2, -1)$, $M(6, -2)$, and $N(4, 2)$. Find the coordinates of the vertices after a 90° rotation about the origin and about each of the points L, M, and N.

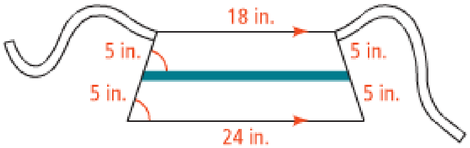
Standardized Test Prep

SAT/ACT

54. What is the image of $(1, -6)$ for a 90° counterclockwise rotation about the origin?

Ⓐ $(6, 1)$ Ⓑ $(-1, 6)$ Ⓒ $(-6, -1)$ Ⓓ $(-1, -6)$

55. The costume crew for your school musical makes aprons like the one shown. If blue ribbon costs \$1.50 per foot, what is the cost of ribbon for six aprons?



Ⓕ \$15.75 Ⓗ \$42.00

Ⓖ \$31.50 Ⓘ \$63.00

56. In $\triangle ABC$, $m\angle A + m\angle B = 84$. Which statement must be true?

Ⓐ $BC > AC$ Ⓑ $AC > BC$ Ⓒ $AB > BC$ Ⓓ $BC > AB$

Short Response

57. Use the following statement: If two lines are parallel, then the lines do not intersect.

a. What are the converse, inverse, and contrapositive of the statement?

b. What is the truth value of each statement you wrote in part (a)? If a statement is false, give a counterexample.

Mixed Review

$\triangle BIG$ has vertices $B(-4, 2)$, $I(0, -3)$, and $G(1, 0)$. Graph $\triangle BIG$ and its reflection image across the given line.

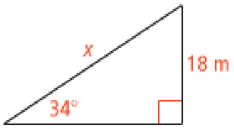
See Lesson 9-2.

58. the y-axis **59.** the x-axis **60.** $x = 4$

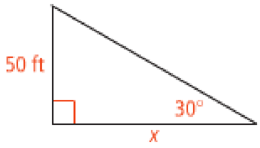
Find the value of x. Round answers to the nearest tenth.

See Lessons 8-2 and 8-3.

61.



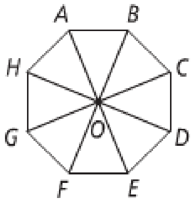
62.



Get Ready! To prepare for Lesson 9-4, do Exercises 63–66.

Point O is the center of the regular octagon. Find the image of the given point or segment for the given rotation or reflection.

See Lessons 9-2 and 9-3.



63. point A; a 90° rotation about center O

64. point H; a 180° rotation about center O

65. $\overline{AB}$; a reflection across $\overleftrightarrow{AE}$

66. $\overline{GH}$; a reflection across $\overleftrightarrow{AE}$

Concept Byte

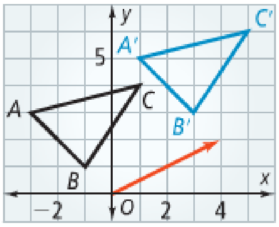
Use With Lesson 9-3

ACTIVITY

Tracing Paper Transformations

In Lesson 9-1, you learned how to describe a translation using variables. In these activities, you will use tracing paper to perform translations, rotations, and reflections. You will also describe certain rotations and reflections using variables.

Activity 1



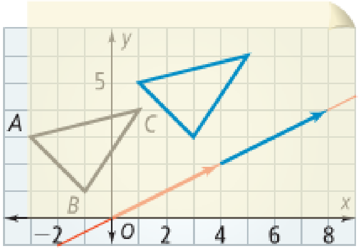
You can use the vector arrow shown in the diagram to represent the translation $(x, y) \rightarrow (x + 4, y + 2)$. The translation shifts $\triangle ABC$ with $A(-3, 3)$, $B(-1, 1)$, and $C(1, 4)$ to $\triangle A'B'C'$ with $A'(1, 5)$, $B'(3, 3)$ and $C'(5, 6)$. You can see this translation using tracing paper as follows:

Step 1 Draw $\triangle ABC$ and the vector arrow on graph paper. Also, show the line containing the arrow.

Step 2 Trace $\triangle ABC$ and the vector arrow.

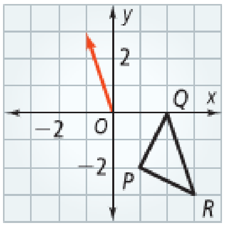
Step 2 Trace $\triangle ABC$ and the vector arrow.

Step 3 Move your tracing of the vector arrow along the vector line until the tail of the tracing is on the head of the original vector arrow. The vertices of your tracing of $\triangle ABC$ should now be at $A'(1, 5)$, $B'(3, 3)$, and $C'(5, 6)$.

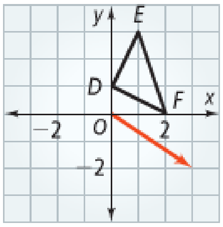


Use tracing paper. Find the translation image of each triangle for the given vector.

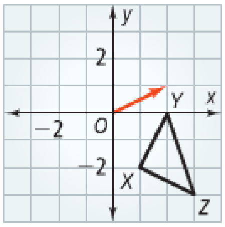
1.



2.



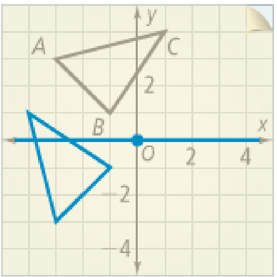
3.



4. Show that the composition of the translation in Question 1 followed by the translation in Question 2 gives you the translation in Question 3.

Activity 2

To rotate a figure 90° about the origin, trace the figure, one axis, and the origin. Then turn your tracing paper counterclockwise, keeping the origin in place and aligning the traced axis with the other original axis.



5. Use $\triangle ABC$ from Activity 1 with $A(-3, 3)$, $B(-1, 1)$, and $C(1, 4)$. What is the image of $\triangle ABC$ for a 90° rotation about the origin?

6. Copy and complete the table. Use tracing paper to find point P', the image of point P, for a rotation of 90° about the origin.

P	P'
(3, 4)	■
(−3, 4)	■
(−3, −4)	■
(3, −4)	■
(3, 0)	■
(0, 4)	■

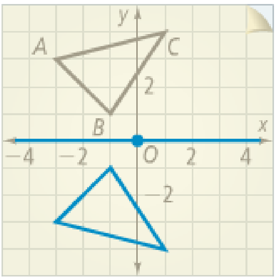
7. Study the pattern in your table. Complete this rule for a rotation of 90° about the origin: $(x, y) \rightarrow$ (■, ■).

8. Test your rule with tracing paper. For $\triangle TRN$ with $T(0, 2)$, $R(3, 0)$, and $N(4, 5)$, a 90° rotation about the origin should result in $\triangle T'R'N'$ with $T'(-2, 0)$, $R'(0, 3)$, and $N'(-5, 4)$. Does it?

9. In parts (a) and (b) below, what should be the result of each composition?
- **a.** a 90° rotation about the origin followed by a 90° rotation about the origin
- **b.** $(x, y) \rightarrow (-y, x)$ followed by $(x, y) \rightarrow (-y, x)$
- **c.** Use tracing paper. Test your conjectures from parts (a) and (b) on $\triangle ABC$ from Activity 1.

Activity 3

To reflect a figure across an axis using tracing paper, trace the figure, the axis, and the origin. Then turn over your tracing paper, keeping the origin in place and aligning the traced axis with the original axis.



10. What is the reflection image of $\triangle ABC$ across the x-axis? Across the y-axis?

11. Copy and complete the table. Use tracing paper to find point P_x and P_y, the reflection images of point P across the x-axis and y-axis, respectively.

P	P_x	P_y
(3, 4)	■	■
(−3, 4)	■	■
(−3, −4)	■	■
(3, −4)	■	■
(3, 0)	■	■
(0, 4)	■	■

12. Study the table. Complete these rules for reflections across the axes.
- **a.** x-axis: $(x, y) \rightarrow$ (■, ■)
- **b.** y-axis: $(x, y) \rightarrow$ (■, ■)

13. Test your rules on $\triangle KLM$ with $K(-2, 3)$, $L(3, 1)$, and $M(1, -2)$.
- **a.** Across the x-axis, $\triangle KLM$ should map to _?_. Does it?
- **b.** Across the y-axis, $\triangle KLM$ should map to _?_. Does it?

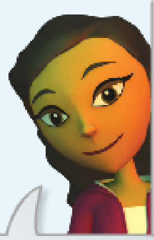
14. In parts (a) and (b) below, what should be the result of each composition?
- **a.** $(x, y) \rightarrow (x, -y)$ followed by $(x, y) \rightarrow (-x, y)$
- **b.** a reflection across the x-axis followed by a reflection across the y-axis
- **c.** Use tracing paper. Test your conjectures from parts (a) and (b) on $\triangle ABC$ from Activity 1.

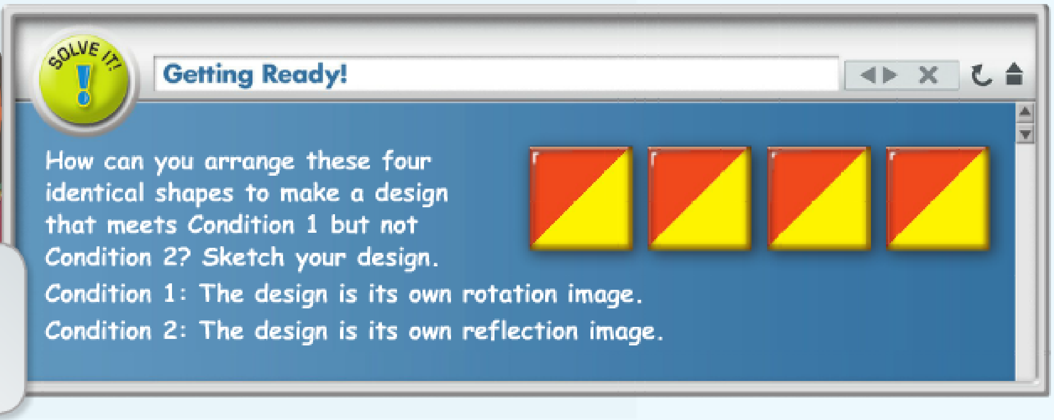
15. Compare the results of Questions 9 and 14. Make a conjecture about the compositions suggested by each.

16. Use tracing paper to find a rule for a reflection across the line $y = x$. Test your rule on $\triangle ABC$ from Activity 1.

9-4 Symmetry

Objective To identify the type of symmetry in a figure


Here's a chance to apply what you know about rotations and reflections.



How can you arrange these four identical shapes to make a design that meets Condition 1 but not Condition 2? Sketch your design.

Condition 1: The design is its own rotation image.

Condition 2: The design is its own reflection image.

In the Solve It, you made a figure that is its own image under a transformation. In this lesson, you will learn to classify figures with this property.

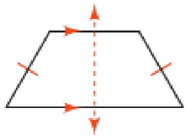
Lesson Vocabulary
- symmetry
- line symmetry
- reflectional symmetry
- line of symmetry
- rotational symmetry
- point symmetry

Essential Understanding Some figures appear unchanged after a reflection across a line or a rotation about a point. Such figures are said to have *symmetry*.

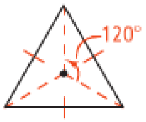
A figure has **symmetry** if there is an isometry that maps the figure onto itself.

take note Key Concept Types of Symmetry

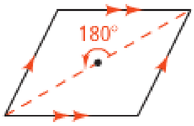
A figure has **line symmetry** or **reflectional symmetry** if there is a reflection for which the figure is its own image. The line of reflection is called a **line of symmetry.** It divides the figure into congruent halves.



A figure has **rotational symmetry** if there is a rotation of 180° or less for which the figure is its own image. The angle of rotation for rotational symmetry is the smallest angle needed for the figure to rotate onto itself.



A figure with 180° rotational symmetry also has **point symmetry.** Each segment joining a point and its 180° rotation image passes through the center of rotation.



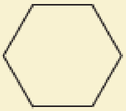
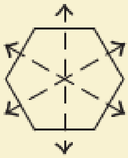
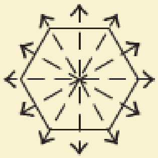
A square, which has both 90° and 180° rotational symmetry, also has point symmetry.

Problem 1 Identifying Lines of Symmetry

How many lines of symmetry does a regular hexagon have?

Plan

How can you see the lines of symmetry? *Draw a diagram* of a regular hexagon and look for the ways it will reflect across a line onto itself.

Think	Write
Sketch a regular hexagon.	
The hexagon reflects onto itself across each line that passes through the midpoints of a pair of parallel sides.	
The hexagon also reflects onto itself across each diagonal that passes through the center of the hexagon.	
Count the lines of symmetry.	A regular hexagon has six lines of symmetry.

Got It? 1. a. Draw a rectangle that is not a square. How many lines of symmetry does your rectangle have?

b. Reasoning Is a median of a triangle a line of symmetry for the triangle? Explain.

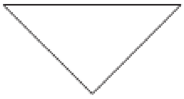
Problem 2 Identifying Rotational Symmetry

Does the figure have rotational symmetry? If so, what is the angle of rotation?

Plan

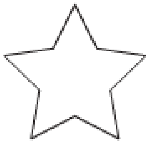
How do you identify rotational symmetry? Look for a possible center point. Think about the angles formed by joining preimage-image pairs to the center. All these angles must be congruent for the figure to have rotational symmetry.

A

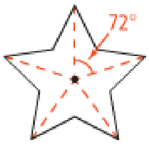


There is no center point about which the triangle will rotate onto itself. This figure does not have rotational symmetry.

B

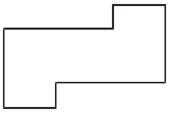


The star has rotational symmetry. The angle of rotation is 72°.



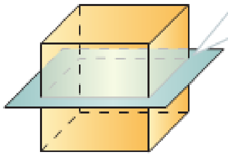
Got It? 2. a. Does the figure at the right have rotational symmetry? If so, what is the angle of rotation?

b. Does the figure have point symmetry? Explain.



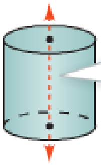
Three-dimensional objects can have various types of symmetry, including *reflectional symmetry in a plane* and *rotational symmetry about a line*.

Reflectional Symmetry



The plane divides the object into congruent halves.

Rotational Symmetry



The object can be rotated about a line so that the image matches the preimage.

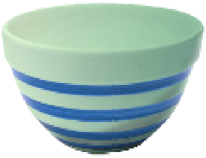
Problem 3 Identifying Symmetry in a Three-Dimensional Object

Symmetric Design **Does the object have reflectional symmetry in a plane, rotational symmetry about a line, or both?**

Think

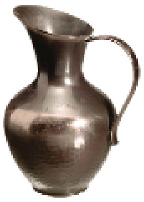
How can you identify symmetry in a 3-D object?
For reflectional symmetry, try to visualize a plane slicing the object. For rotational symmetry, try to imagine rotating the object about a line.

A



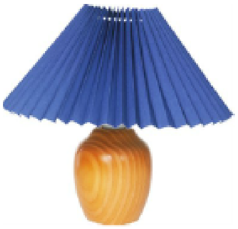
The bowl has both reflectional symmetry in a plane and rotational symmetry about a line.

B



The pitcher has reflectional symmetry in a plane.

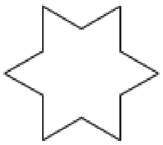
Got It? **3.** Does the lampshade have reflectional symmetry in a plane, rotational symmetry about a line, or both?



Lesson Check

Do you know HOW?

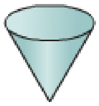
Use the figure below for Exercises 1 and 2.



1. Does the figure have line symmetry? If so, copy the figure and draw its line(s) of symmetry.

2. Does the figure have rotational symmetry? If so, what is the angle of rotation?

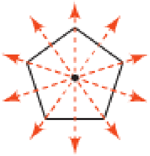
3. What type(s) of symmetry does the three-dimensional figure have?



Do you UNDERSTAND?

4. **Vocabulary** If a figure has point symmetry, must it also have rotational symmetry? Explain.

5. **Error Analysis** Your friend thinks that the regular pentagon in the diagram at the right has 10 lines of symmetry. Explain and correct your friend's error.



6. **Open-Ended** The word CHECKBOOK has a horizontal line of symmetry. Find two other words for which this is true.

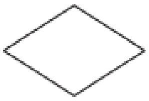
Practice and Problem-Solving Exercises

A Practice

Tell what type(s) of symmetry each figure has. If it has line symmetry, sketch the figure and the line(s) of symmetry. If it has rotational symmetry, tell the angle of rotation.

See Problems 1 and 2.

7.



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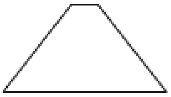

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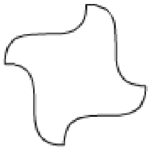

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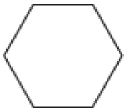

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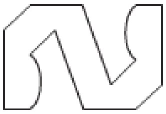
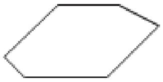


12.



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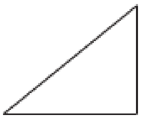

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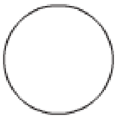

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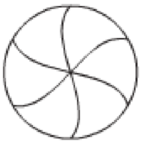

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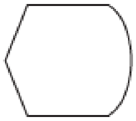
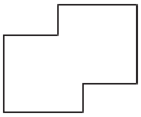


17.



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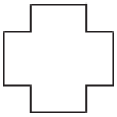


Determine how many lines of symmetry each type of quadrilateral has. Include a sketch to support your answer.

19. rhombus

20. kite

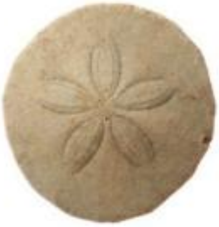
21. square

22. parallelogram

Tell whether the subject of the photo has reflectional symmetry in a plane, rotational symmetry about a line, or both.

See Problem 3.

23.



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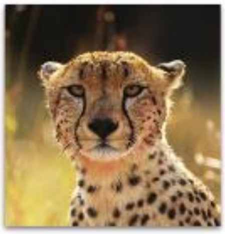


B Apply

25. Alphabets Use the letters of the alphabets below.

English: ABCDEFGHIJKLMNOPQRSTUVWXYZ

Greek: ΒΓΔΕΖΗΘΙΚΛΜΝΞΟΠΡΣΤΥΦΧΨΩ

a. Copy the table. Classify the letters of the alphabets. You will list some letters in more than one category.

b. Which alphabet has more symmetrical letters? Explain.

Alphabet Symmetry

Language	Type of Symmetry		
	Horizontal Line	Vertical Line	Point
English	■	■	■
Greek	■	■	■

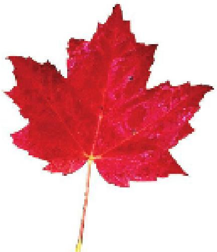
26. **Open-Ended** If you stack the letters of MATH vertically, you can find a vertical line of symmetry. Find two other words for which this is true.

27. **Think About a Plan** A quadrilateral with vertices at (1, 5) and (−2, −3) has point symmetry about the origin. Show that the quadrilateral is a parallelogram.

- How can you use point symmetry to find the other vertices?
- How can you show a quadrilateral is a parallelogram?

Tell what type(s) of symmetry each figure has. For line symmetry, sketch the figure and the line(s) of symmetry. For rotational symmetry, tell the angle of rotation.

28.



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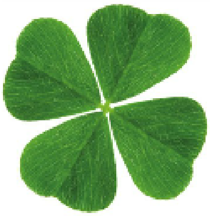


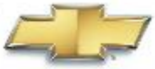
30. **Reasoning** Is the line that contains the bisector of an angle also a line of symmetry of the angle? Explain.

31. **Reasoning** Is the line that contains the bisector of an angle of a triangle also a line of symmetry of the triangle? Explain.

32. **Open-Ended** The equation $\frac{10}{10} - 1 = 0 \div \frac{83}{83}$ is not only true, but also symmetrical (horizontally). Write four other equations or inequalities that are both true and symmetrical.

Logos **Describe the types of symmetry, if any, of each automobile logo.**

33.



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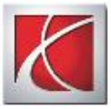

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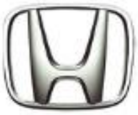

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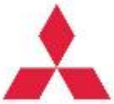
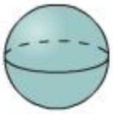


Tell whether each three-dimensional object has reflectional symmetry in a plane, rotational symmetry about a line, or both.

37.



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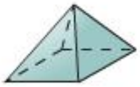

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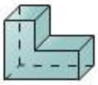

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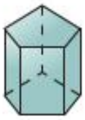


Coordinate Geometry A figure that has a vertex at (3, 4) has the given line of symmetry. Tell the coordinates of another vertex of the figure.

41. the y-axis

42. the x-axis

43. the line $y = x$

Challenge

Coordinate Geometry Graph each equation and describe its symmetry.

44. $y = x^2$

45. $y = (x + 2)^2$

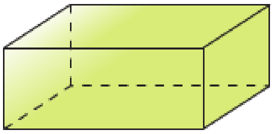
46. $y = x^3$

47. $y = |x|$

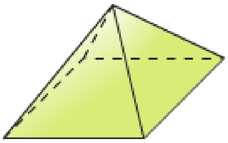
48. $x^2 + y^2 = 9$

For each three-dimensional figure, draw a net that has point symmetry and a net that has 1, 2, or 4 lines of symmetry.

49.



50.



Square pyramid

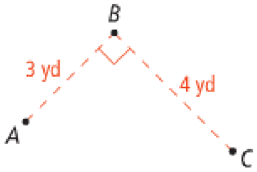
Standardized Test Prep

GRIDDED RESPONSE

SAT/ACT

51. What is the smallest angle, in degrees, through which you can rotate a regular hexagon onto itself?

52. You place a sprinkler so that it is equidistant from three rose bushes at points A, B, and C. How many feet is the sprinkler from A?



53. $\triangle STU$ has vertices $S(1, 2)$, $T(0, 5)$, and $U(-8, 0)$. What is the x-coordinate of S after a 270° rotation about the origin?

54. The diagonals of rectangle $PQRS$ intersect at O. $PO = 2x - 5$ and $OR = 7 - x$. What is the length of $\overline{QS}$?

55. The length of the hypotenuse of a 45°-45°-90° triangle is 55 in. What is the length of one of its legs to the nearest tenth of an inch?

Mixed Review

56. Which capital letters of the alphabet are rotation images of themselves? Draw each letter and give an angle of rotation ($< 360°$).

See Lesson 9-3.

57. Three vertices of an isosceles trapezoid are $(-2, 1)$, $(1, 4)$, and $(4, 4)$. Find all possible coordinates for the fourth vertex.

See Lesson 6-7.

Get Ready! **To prepare for Lessons 9-5, do Exercises 58–60.**

Determine the scale drawing dimensions of each room using a scale of $\frac{1}{4}$ in. = 1 ft.

See Lesson 7-2.

58. kitchen: 12 ft by 16 ft

59. bedroom: 8 ft by 10 ft

60. laundry room: 6 ft by 9 ft

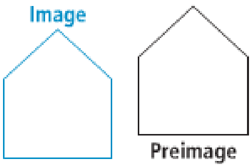
9 Mid-Chapter Quiz

MathXL® for School
Go to PowerGeometry.com

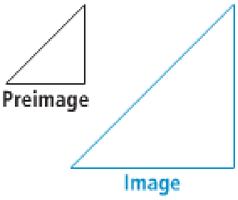
Do you know HOW?

Tell whether the transformation appears to be an isometry. Explain.

1.



2.



3. What rule describes the translation 5 units left and 10 units up?

4. Describe the translation $(x, y) \rightarrow (x - 3, y + 5)$ in words.

5. Find a translation that has the same effect as the following composition of translations.
$(x, y) \rightarrow (x + 7, y - 2)$ followed by
$(x, y) \rightarrow (x - 3, y + 2)$

Find the reflection image of (5, −3) across each line.

6. the x-axis

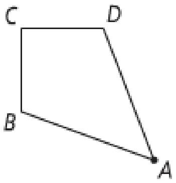
7. y-axis

8. $\triangle WXY$ has vertices $W(-4, 1)$, $X(2, -7)$, and $Y(0, -3)$. What are the vertices of the image of $\triangle WXY$ for the translation $(x, y) \rightarrow (x - 3, y + 5)$?

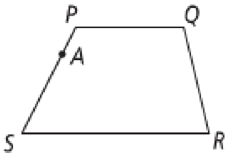
9. $\triangle ABC$ has vertices $A(-1, 4)$, $B(2, 0)$, and $C(4, 3)$. Graph the reflection image of $\triangle ABC$ across the line $x = -1$.

Copy each figure and point A. Draw the image of each figure for the given angle of rotation about A. Use prime notation to label the vertices of the image.

10. 40°



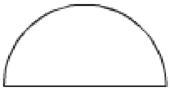
11. 90°



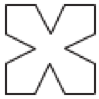
12. How many lines of symmetry does a regular pentagon have? Include a sketch in your answer.

Tell what type(s) of symmetry each figure has. For line symmetry, sketch the image and the line(s) of symmetry. For rotational symmetry, state the angle of rotation.

13.



14.



Do you UNDERSTAND?

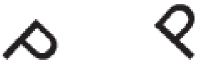
15. Reasoning The point $(5, -9)$ is the image under the translation $(x, y) \rightarrow (x - 3, y + 2)$. What is its preimage?

16. Reasoning The point $T(5, -1)$ is reflected first across the x-axis and then across the y-axis. What is the distance between the image and the preimage? Explain.

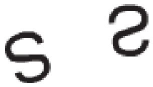
17. Coordinate Geometry The point $L(a, b)$ in Quadrant I is rotated 90° about the origin and then reflected across the y-axis. What are the coordinates of the image of L?

The two letters in each pair are congruent. Does one figure appear to be a translation image, a reflection image, or a rotation image of the other?

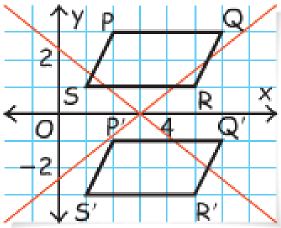
18.



19.



20. Error Analysis Your friend draws the diagram at the right to show the reflection of ▱$PQRS$ across the x-axis. Explain and correct your friend's error.



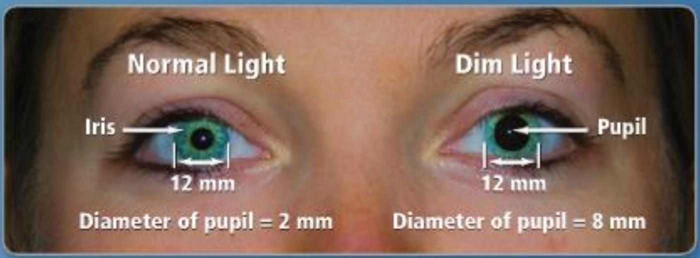
9-5 Dilations

Objective To understand dilation images of figures

Look for what changes and what does not change.

SOLVE IT! **Getting Ready!**

The pupil is the opening in the iris that lets light into the eye. Depending on the amount of light available, the size of the pupil changes.



Think of extending the definition of similar polygons you learned in Chapter 7 to apply to curved figures. Is the pupil in dim light similar to the pupil in normal light? Is the iris in dim light similar to the iris in normal light? Explain.

Lesson Vocabulary
- dilation
- center of dilation
- scale factor of a dilation
- enlargement
- reduction

In the Solve It, you looked at how the pupil of an eye changes in size, or *dilates*. In this lesson, you will learn how to dilate geometric figures.

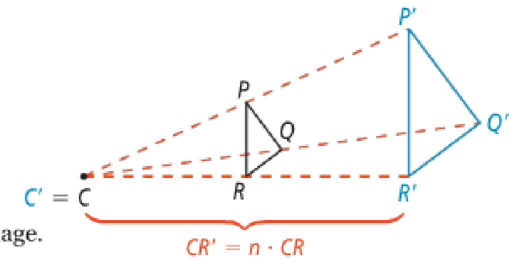
Essential Understanding You can use a scale factor to make a larger or smaller copy of a figure that is also similar to the original figure.

take note **Key Concept** Dilation

A **dilation** with **center** C and **scale factor** n, $n > 0$, is a transformation with these two properties:

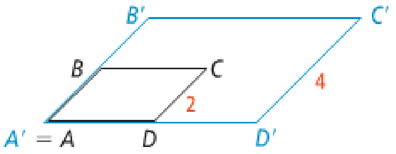
- The image of C is itself (that is, $C' = C$).
- For any other point R, R' is on $\overrightarrow{CR}$ and $CR' = n \cdot CR$, or $n = \frac{CR'}{CR}$.

The image of a dilation is similar to its preimage.

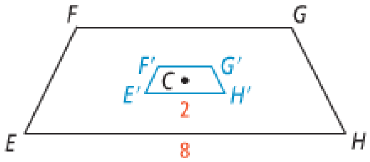


You can show that the scale factor of a dilation has the same value as the scale factor of the similar figures (preimage and image), with the image length in the numerator. For the figure shown on page 575, $n = \frac{CR'}{CR} = \frac{R'P'}{RP}$.

A dilation is an **enlargement** if the scale factor is greater than 1. The dilation is a **reduction** if the scale factor is between 0 and 1.



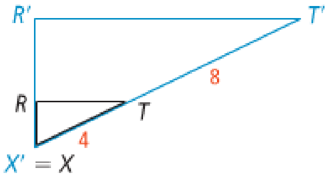
Enlargement
center A, scale factor 2



Reduction
center C, scale factor $\frac{1}{4}$

Problem 1 Finding a Scale Factor

Multiple Choice $\triangle X'T'R'$ **is a dilation image of** $\triangle XTR$**. The center of dilation is** X**. Is the dilation an enlargement or a reduction? What is the scale factor of the dilation?**



Ⓐ enlargement; scale factor 2

Ⓑ enlargement; scale factor 3

Ⓒ reduction; scale factor $\frac{1}{3}$

Ⓓ reduction; scale factor 3

The image is larger than the preimage, so the dilation is an enlargement. Use the ratio of the lengths of corresponding sides to find the scale factor.

$$n = \frac{X'T'}{XT} = \frac{4 + 8}{4} = \frac{12}{4} = 3$$

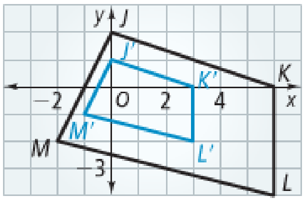
$\triangle X'T'R'$ is an enlargement of $\triangle XTR$, with a scale factor of 3. The correct answer is B.

Think

Why is the scale factor not $\frac{4}{12}$, or $\frac{1}{3}$?

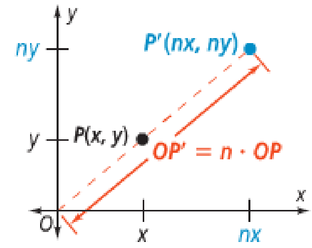
The scale factor of a dilation always has the image length (or the distance between a point on the image and the center) in the numerator.

Got It? **1.** $J'K'L'M'$ is a dilation image of $JKLM$. The center of dilation is O. Is the dilation an enlargement or a reduction? What is the scale factor of the dilation?



In Got It 1, you looked at a dilation of a figure drawn in the coordinate plane. In this book, all dilations of figures in the coordinate plane have the origin as the center of dilation. So you can find the dilation image of a point $P(x, y)$ by multiplying the coordinates of P by the scale factor n.

$$P(x, y) \rightarrow P'(nx, ny)$$

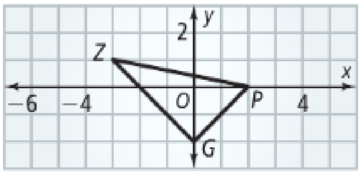


Problem 2 Finding a Dilation Image

Think

Will the vertices of the triangle move closer to (0, 0) or farther from (0, 0)?
The scale factor is 2, so the dilation is an enlargement. The vertices will move farther from (0, 0).

What are the images of the vertices of $\triangle PZG$ for a dilation with center (0, 0) and scale factor 2? Graph the image of $\triangle PZG$.



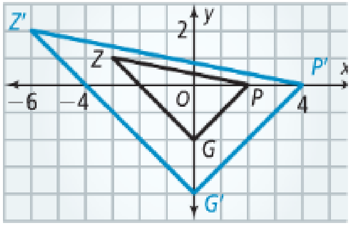
Identify the coordinates of each vertex. The center of dilation is the origin and the scale factor is 2, so use the dilation rule $(x, y) \rightarrow (2x, 2y)$.

$P(2, 0) \rightarrow (2 \cdot 2, 2 \cdot 0)$, or $P'(4, 0)$.

$Z(-3, 1) \rightarrow (2 \cdot (-3), 2 \cdot 1)$, or $Z'(-6, 2)$.

$G(0, -2) \rightarrow (2 \cdot 0, 2 \cdot (-2))$, or $G'(0, -4)$.

To graph the image of $\triangle PZG$, graph P', Z', and G'. Then draw $\triangle P'Z'G'$.



Got It? **2. a.** What are the images of the vertices of $\triangle PZG$ for a dilation with center (0, 0) and scale factor $\frac{1}{2}$?

b. Reasoning In Problem 2, how can you prove $\triangle P'Z'G' \sim \triangle PZG$ using a postulate or theorem from Chapter 7?

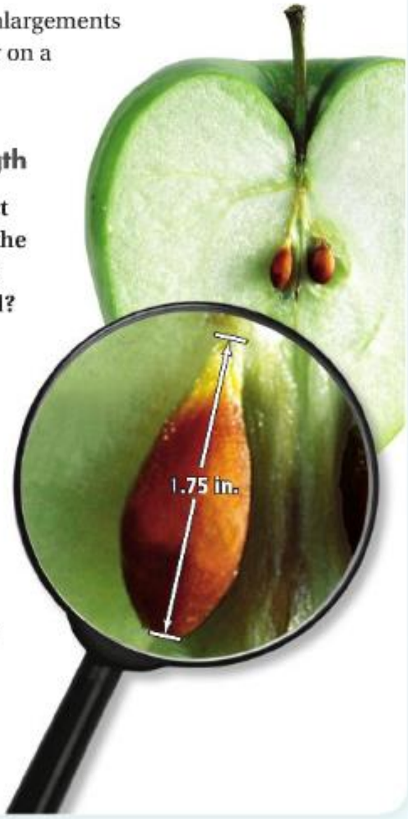
Dilations and scale factors help you understand real-world enlargements and reductions, such as images seen through a microscope or on a computer screen.

Problem 3 Using a Scale Factor to Find a Length

Think

What does a scale factor of 7 tell you?
A scale factor of 7 tells you that the ratio of the image length to the actual length is 7, or $\frac{\text{image length}}{\text{actual length}} = 7$.

Biology A magnifying glass shows you an image of an object that is 7 times the object's actual size. So the scale factor of the enlargement is 7. The photo shows an apple seed under this magnifying glass. What is the actual length of the apple seed?



$1.75 = 7 \cdot p$ image length = scale factor · actual length

$0.25 = p$ Divide each side by 7.

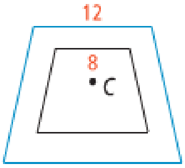
The actual length of the apple seed is 0.25 in.

Got It? **3.** The height of a document on your computer screen is 20.4 cm. When you change the zoom setting on your screen from 100% to 25%, the new image of your document is a dilation of the previous image with scale factor 0.25. What is the height of the new image?

Lesson Check

Do you know HOW?

1. The blue figure is a dilation image of the black figure with center of dilation C. Is the dilation an enlargement or a reduction? What is the scale factor of the dilation?



A dilation has center (0, 0). Find the image of each point for the given scale factor.

2. $D(1, -5)$; 2

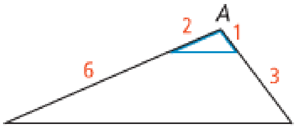
3. $T(0, 6)$; $\frac{1}{3}$

4. $M(0, 0)$; 10

Do you UNDERSTAND?

5. Vocabulary Describe the scale factor of a reduction.

6. Error Analysis The blue figure is a dilation image of the black figure for a dilation with center A.



Two students made errors when asked to find the scale factor. Explain and correct their errors.

A. ~~$n = \frac{2}{6} = \frac{1}{3}$~~

B. ~~$n = \frac{4}{1} = 4$~~

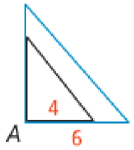
Practice and Problem-Solving Exercises

A Practice

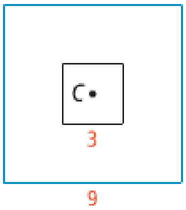
The blue figure is a dilation image of the black figure. The labeled point is the center of dilation. Tell whether the dilation is an enlargement or a reduction. Then find the scale factor of the dilation.

See Problem 1.

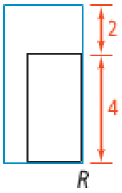
7.



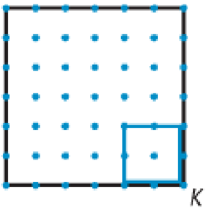
8.



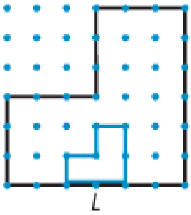
9.



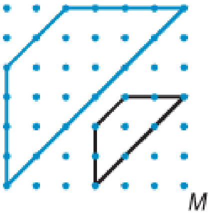
10.



11.



12.



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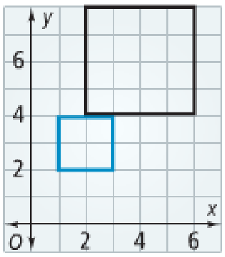
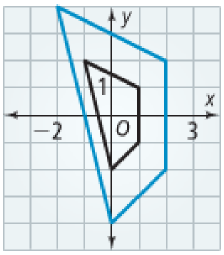
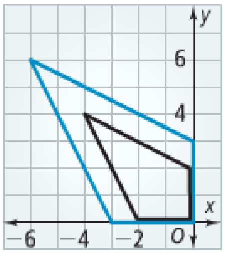


14.



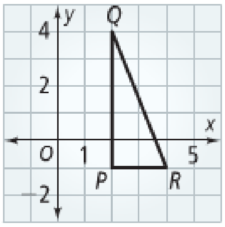
15.



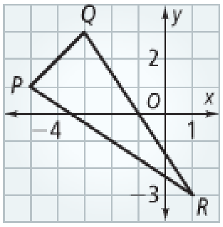
Find the images of the vertices of $\triangle PQR$ for a dilation with center (0, 0) and the given scale factor. Graph the image.

See Problem 2.

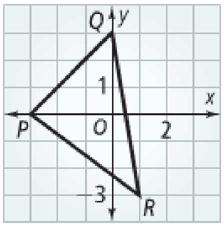
16. scale factor 3



17. scale factor 10



18. scale factor $\frac{3}{4}$



Magnification **You look at each object described in Exercises 19–22 under a magnifying glass. Find the actual dimension of each object.**

See Problem 3.

19. The image of a button is 5 times the button's actual size and has a diameter of 6 cm.

20. The image of a pinhead is 8 times the pinhead's actual size and has a width of 1.36 cm.

21. The image of an ant is 7 times the ant's actual size and has a length of 1.4 cm.

22. The image of a capital letter N is 6 times the letter's actual size and has a height of 1.68 cm.

B Apply

A dilation has center (0, 0). Find the image of each point for the given scale factor.

23. $L(-3, 0); 5$

24. $N(-4, 7); 0.2$

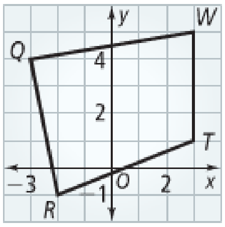
25. $A(-6, 2); 1.5$

26. $F(3, -2); \frac{1}{3}$

27. $B\left(\frac{5}{4}, -\frac{3}{2}\right); \frac{1}{10}$

28. $Q\left(6, \frac{\sqrt{3}}{2}\right); \sqrt{6}$

Use the graph at the right. Find the vertices of the image of $QRTW$ for a dilation with center (0, 0) and the given scale factor.



29. $\frac{1}{4}$

30. 0.6

31. 0.9

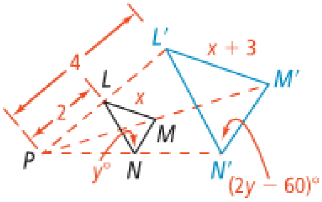
32. 10

33. 100

34. **Compare and Contrast** Compare the definition of scale factor of a dilation to the definition of scale factor of two similar polygons. How are they alike? How are they different?

35. **Think About a Plan** The diagram at the right shows $\triangle LMN$ and its image $\triangle L'M'N'$ for a dilation with center P. Find the values of x and y. Explain your reasoning.

- What is the relationship between $\triangle LMN$ and $\triangle L'M'N'$?
- What is the scale factor of the dilation?
- Which variable can you find using the scale factor?

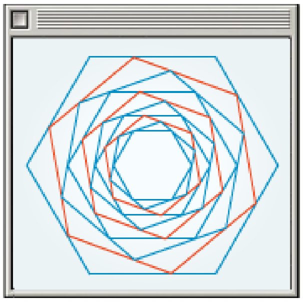


36. **Writing** An equilateral triangle has 4-in. sides. Describe its image for a dilation with center at one of the triangle's vertices and scale factor 2.5.

Coordinate Geometry **Graph $MNPQ$ and its image $M'N'P'Q'$ for a dilation with center $(0, 0)$ and the given scale factor.**

37. $M(1, 3)$, $N(-3, 3)$, $P(-5, -3)$, $Q(-1, -3)$; 3

38. $M(2, 6)$, $N(-4, 10)$, $P(-4, -8)$, $Q(-2, -12)$; $\frac{1}{4}$



39. Open-Ended Use the dilation command in geometry software or drawing software to create a design that involves repeated dilations, such as the one shown at the right. The software will prompt you to specify a center of dilation and a scale factor. Print your design and color it. Feel free to use other transformations along with dilations.

40. Copy Reduction Your picture of your family crest is 4.5 in. wide. You need a reduced copy for the front page of the family newsletter. The copy must fit in a space 1.8 in. wide. What scale factor should you use on the copy machine to adjust the size of your picture of the crest?

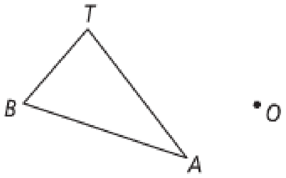
A dilation maps $\triangle HIJ$ onto $\triangle H'I'J'$. Find the missing values.

41. $HI = 8$ in. $H'I' = 16$ in.
$IJ = 5$ in. $I'J' =$ ■ in.
$HJ = 6$ in. $H'J' =$ ■ in.

42. $HI = 7$ cm $H'I' = 5.25$ cm
$IJ = 7$ cm $I'J' =$ ■ cm
$HJ =$ ■ cm $H'J' = 9$ cm

43. $HI =$ ■ ft $H'I' = 8$ ft
$IJ = 30$ ft $I'J' =$ ■ ft
$HJ = 24$ ft $H'J' = 6$ ft

Copy $\triangle TBA$ and point O for each of Exercises 44–47. Draw the dilation image $\triangle T'B'A'$ for the given center and scale factor.



44. center O, scale factor 2

45. center B, scale factor 3

46. center T, scale factor $\frac{1}{3}$

47. center O, scale factor $\frac{1}{2}$

48. Reasoning You are given $\overline{AB}$ and its dilation image $\overline{A'B'}$ with A, B, A', and B' noncollinear. Explain how to find the center of dilation and scale factor.

Reasoning **Write *true* or *false* for Exercises 49–52. Explain your answers.**

49. A dilation is an isometry.

50. A dilation with a scale factor greater than 1 is a reduction.

51. For a dilation, corresponding angles of the image and preimage are congruent.

52. A dilation image cannot have any points in common with its preimage.

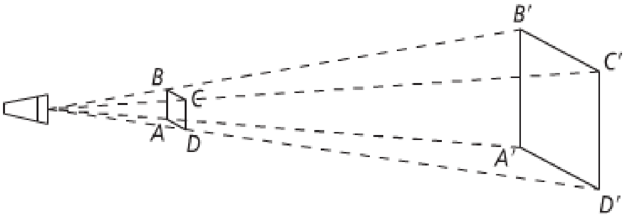
Challenge

Coordinate Geometry **In the coordinate plane, you can extend dilations to include scale factors that are negative numbers. For Exercises 53 and 54, use $\triangle PQR$ with vertices $P(1, 2)$, $Q(3, 4)$, and $R(4, 1)$.**

53. Graph $\triangle PQR$ and its image for a dilation centered at $(0, 0)$ with scale factor -3.

54. a. Graph $\triangle PQR$ and its image for a dilation centered at $(0, 0)$ with scale factor -1.
b. Explain why the dilation in part (a) may be called a *reflection through a point*. Extend your explanation to a new definition of point symmetry.

55. **Shadows** A flashlight projects an image of rectangle $ABCD$ on a wall so that each vertex of $ABCD$ is 3 ft away from the corresponding vertex of $A'B'C'D'$. The length of $\overline{AB}$ is 3 in. The length of $\overline{A'B'}$ is 1 ft. How far from each vertex of $ABCD$ is the light?



Standardized Test Prep

SAT/ACT

56. A dilation maps $\triangle CDE$ onto $\triangle C'D'E'$. If $CD = 7.5$ ft, $CE = 15$ ft, $D'E' = 3.25$ ft, and $C'D' = 2.5$ ft, what is DE?

Ⓐ 1.08 ft Ⓑ 5 ft Ⓒ 9.75 ft Ⓓ 19 ft

57. You want to prove indirectly that the diagonals of a rectangle are congruent. As the first step of your proof, what should you assume?

Ⓕ A quadrilateral is not a rectangle.

Ⓖ The diagonals of a rectangle are not congruent.

Ⓗ A quadrilateral has no diagonals.

Ⓘ The diagonals of a rectangle are congruent.

58. Which word can describe a kite?

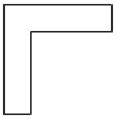
Ⓐ equilateral Ⓑ equiangular Ⓒ convex Ⓓ scalene

Short Response

59. Use the figure at the right to answer the questions below.

a. Does the figure have rotational symmetry? If so, identify the angle of rotation.

b. Does the figure have reflectional symmetry? If so, how many lines of symmetry does it have?



Mixed Review

Coordinate Geometry **A figure with a vertex at $(-2, 7)$ has the given type of symmetry. State the coordinates of another vertex of the figure.**

See Lesson 9-4.

60. point symmetry centered at the origin

61. line symmetry across the y-axis

Get Ready! **To prepare for Lesson 9-6, do Exercises 62–64.**

Given points $R(-1, 1)$, $S(-4, 3)$, and $T(-2, 5)$, graph $\triangle RST$ and its reflection image across each line.

See Lesson 9-2.

62. the y-axis

63. the x-axis

64. $y = 1$

Concept Byte

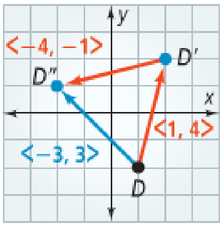
Use With Lesson 9-5

EXTENSION

Transformations Using Vectors and Matrices

In Lesson 8-5, you learned that vectors are quantities that have both direction and magnitude. A translation moves all the points of a figure in the same direction and the same distance, so you can use a vector to describe a translation.

Example 1



A **Find the image of $D(1, -2)$ under the translation described by the vector $\langle 1, 4\rangle$.**

$\langle 1, 4\rangle$ is a translation 1 unit right and 4 units up, the same as $(x, y) \rightarrow (x + 1, y + 4)$.

$D(1, -2)$ translates to $(1 + 1, -2 + 4)$, or $(2, 2)$.

B **Find the image of $D(1, -2)$ under the composition of translations $\langle 1, 4\rangle$ followed by $\langle -4, -1\rangle$.**

Use vector addition: $\langle 1, 4\rangle + \langle -4, -1\rangle = \langle 1 + (-4), 4 + (-1)\rangle = \langle -3, 3\rangle$.

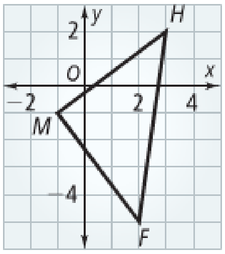
$D(1, -2)$ translates to $(1 - 3, -2 + 3)$, or $(-2, 1)$.

You can use a matrix to find the transformation images of several points at once. A *matrix* is a rectangular arrangement of numbers in rows and columns.

$$\begin{bmatrix} -1 & 4 & 9 \\ 3 & -5 & 7 \end{bmatrix}$$

The matrix at the right is called a 2×3 (two-by-three) matrix because it has 2 rows and 3 columns. Each entry in a matrix is an *element*. This matrix has 6 elements.

Example 2



Use matrices to find the image of $\triangle MFH$ under the translation $\langle 4, -5\rangle$.

$$\begin{array}{rc} & \begin{array}{ccc} M & F & H \end{array} \\ \begin{array}{r} x\text{-coordinate} \\ y\text{-coordinate} \end{array} & \begin{bmatrix} -1 & 2 & 3 \\ -1 & -5 & 2 \end{bmatrix} \end{array}$$

Write a matrix for $\triangle MFH$ made up of the three vertices.

$$\begin{bmatrix} 4 & 4 & 4 \\ -5 & -5 & -5 \end{bmatrix}$$

Write the translation matrix with the translation vector three times.

$$\begin{bmatrix} -1 & 2 & 3 \\ -1 & -5 & 2 \end{bmatrix} + \begin{bmatrix} 4 & 4 & 4 \\ -5 & -5 & -5 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 7 \\ -6 & -10 & -3 \end{bmatrix}$$

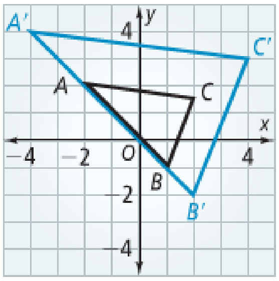
Add the matrices.

The image of $\triangle MFH$ is $\triangle M'F'H'$ with $M'(3, -6)$, $F'(6, -10)$, and $H'(7, -3)$.

To find the image of a triangle for a dilation centered at the origin, multiply each element in the vertex matrix for the triangle by the scale factor. Multiplying a matrix by a number in this way, as with vectors, is called scalar multiplication.

Example 3

$\triangle ABC$ has vertices $A(-2, 2)$, $B(1, -1)$, and $C\left(2, 1\frac{1}{2}\right)$. Use scalar multiplication to find the image of $\triangle ABC$ for a dilation with center $(0, 0)$ and scale factor 2. Graph $\triangle ABC$ and its enlargement.



$$\begin{array}{r c} & \begin{matrix} A & B & C \end{matrix} \\ \text{x-coordinate} \\ \text{y-coordinate} & \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 1\frac{1}{2} \end{bmatrix} \end{array}$$

Write a matrix for $\triangle ABC$.

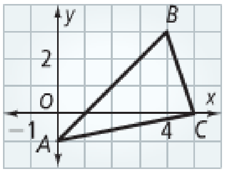
$$2 \cdot \begin{bmatrix} -2 & 1 & 2 \\ 2 & -1 & 1\frac{1}{2} \end{bmatrix} = \begin{bmatrix} -4 & 2 & 4 \\ 4 & -2 & 3 \end{bmatrix}$$

Multiply each element of the matrix by the scale factor 2.

The vertices of the enlargement are $A'(-4, 4)$, $B'(2, -2)$, and $C'(4, 3)$.

Exercises

Find the image of each figure under the translation described by the given vector.



1. $\triangle ABC$ in the graph at the right; vector $= \langle 4, -1 \rangle$

2. quadrilateral $DEFG$ with $D(-4, 1)$, $E(1, 1)$, $F(1, -2)$, $G(0, -2)$; vector $= \langle 2, 0 \rangle$

3. quadrilateral $JKLM$ with $J(-2, 0)$, $K(2, 2)$, $L(3, -2)$, $M(-3, -2)$; vector $= \langle -3, -2 \rangle$

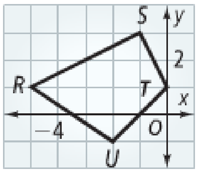
Find a single translation that has the same effect as each composition of translations.

4. $\langle -2, 3 \rangle$ followed by $\langle 1, 5 \rangle$

5. $\langle -6, 0 \rangle$ followed by $\langle -4, 7 \rangle$

6. $\langle 3, -4 \rangle$ followed by $\langle -4, 3 \rangle$

Use matrices to find the image of each polygon under the given translation.



7. quadrilateral $RSTU$ in the graph at the right; translation $= \langle 2, -1 \rangle$

8. $\triangle HJK$ with $H(-2, 2)$, $J(4, 2)$, $K(-1, -2)$; translation $= \langle 4, -3 \rangle$

9. ▱$NCLE$ with $N(-3, 1)$, $C(-4, -1)$, $L(0, -3)$, $E(1, -1)$; translation $= \langle 0, 2 \rangle$

Use a matrix and scalar multiplication. Find the image of the triangle with the given vertices for a dilation with center (0, 0) and the given scale factor. Graph the triangle and its image.

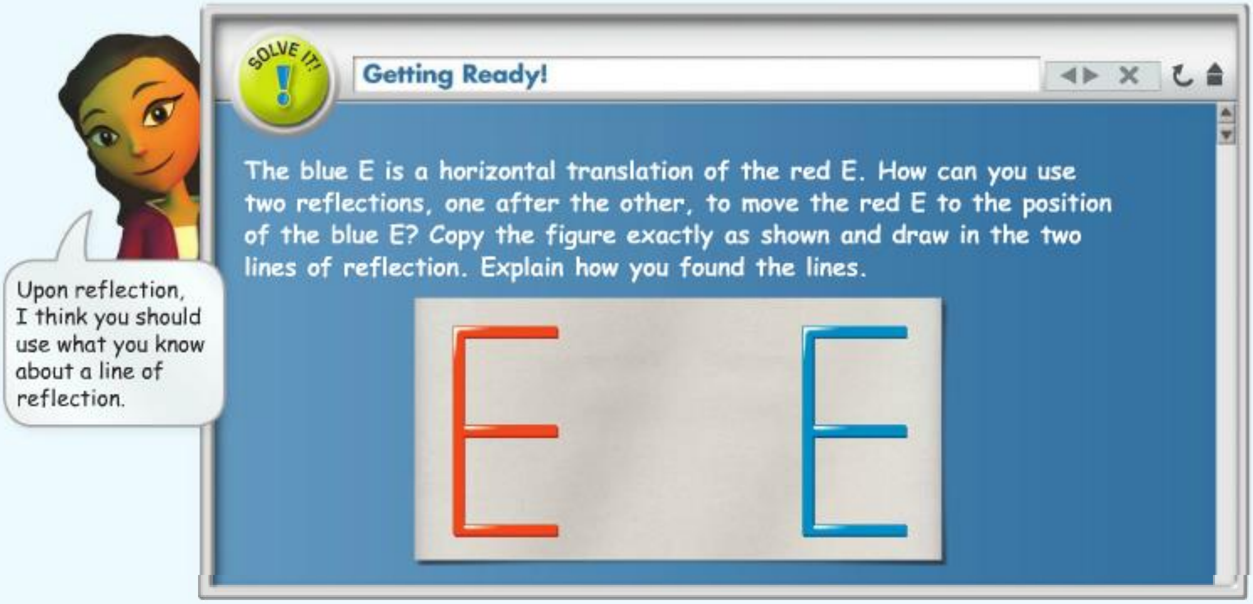
10. $A(0, 0)$, $B(5, 3)$, $C(2, -1)$; 3

11. $R(5, 1)$, $S(-1, -2)$, $T(4, -3)$; $1\frac{1}{2}$

12. $X(-3, 2)$, $Y(-3, -2)$, $Z(2, -2)$; $\frac{1}{3}$

9-6 Compositions of Reflections

Objectives To find compositions of reflections, including glide reflections
To classify isometries


SOLVE IT!
Upon reflection, I think you should use what you know about a line of reflection.

Getting Ready!

The blue E is a horizontal translation of the red E. How can you use two reflections, one after the other, to move the red E to the position of the blue E? Copy the figure exactly as shown and draw in the two lines of reflection. Explain how you found the lines.

Lesson Vocabulary
- glide reflection

In the Solve It, you looked for a way to use two reflections to produce the same image as a given horizontal translation. In this lesson you will see that any isometry can be expressed as a composition of reflections.

Essential Understanding If two figures in a plane are congruent, you can map one onto the other using a composition of reflections.

The theorems in this lesson lead to the fact stated above. Complete proofs of these theorems are beyond the scope of this course, but Problems 1 and 2 suggest approaches to the proofs of Theorems 9-1 and 9-2. Theorem 9-2 is the converse of Theorem 9-1.

take note

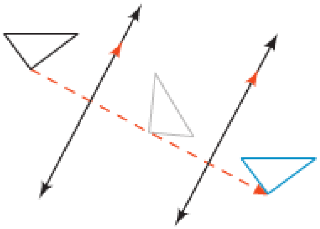
Theorem 9-1

A translation or rotation is a composition of two reflections.

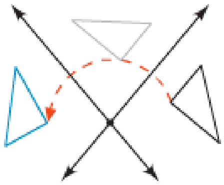
take note

Theorem 9-2

A composition of reflections across two parallel lines is a translation.

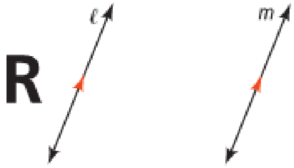


A composition of reflections across two intersecting lines is a rotation.

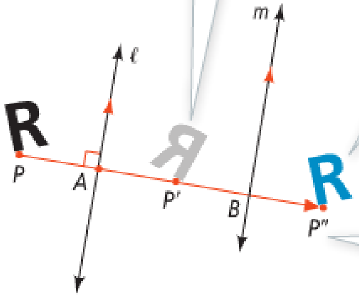


Problem 1 Composing Reflections Across Parallel Lines

What is the image of R reflected first across line ℓ and then across line m? What are the direction and distance of the resulting translation?



As you do the two reflections, keep track of the distance moved by a point P of the preimage.



Step 1 Reflect R across ℓ. $PA = AP'$, so $PP' = 2AP'$.

Step 2 Reflect the image across m. $P'B = BP''$, so $P'P'' = 2P'B$.

P moved a total distance of $2AP' + 2P'B$, or $2AB$.

Think

How do you know that $PA = AP'$, $P'B = BP''$, and $\overleftrightarrow{AB} \perp \ell$?
All three statements are true by the definition of reflection across a line.

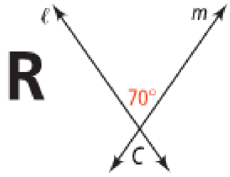
The red arrow shows the direction of the translation, so the direction of the translation is determined by a line perpendicular to ℓ and m. The total distance P moved is $2 \cdot AB$. Because $\overleftrightarrow{AB} \perp \ell$, AB is the distance between ℓ and m. The distance of the translation is twice the distance between ℓ and m.

Got It? **1. a.** Draw parallel lines ℓ and m as in Problem 1. Draw R between ℓ and m. What is the image of R reflected first across line ℓ and then across line m? What are the direction and distance of the resulting translation?

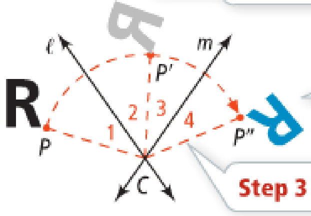
b. **Reasoning** Use the results of part (a) and Problem 1. Make a conjecture about the direction and distance of any translation that is the result of a composition of reflections across two parallel lines.

Problem 2 Composing Reflections Across Intersecting Lines

Lines ℓ and m intersect at point C and form an acute angle that measures 70. What is the image of R reflected first across line ℓ and then across line m? What are the center of rotation and the angle of rotation for the resulting rotation?



After you do the two reflections, follow the path of a point P of the preimage.



Step 1 Reflect R across ℓ.

Step 2 Reflect the image across m.

Step 3 Draw the angles formed by joining P, P', and P'' to C.

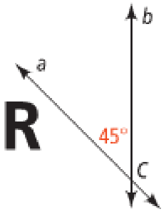
Think

How do you show that $m\angle 1 = m\angle 2$?
If you draw $\overline{PP'}$ and label its intersection point with line ℓ as A, you can prove $\triangle APC \cong \triangle AP'C$ by SSS.

R is rotated clockwise about the intersection point of the lines. The center of rotation is C. You know that $m\angle 2 + m\angle 3 = 70$. You can use the definition of reflection to show that $m\angle 1 = m\angle 2$ and $m\angle 3 = m\angle 4$. So, $m\angle 1 + m\angle 2 + m\angle 3 + m\angle 4 = 140$. The angle of rotation is 140° clockwise.

Got It? **2. a.** Use the diagram at the right. What is the image of R reflected first across line a and then across line b? What are the center of rotation and the angle of rotation for the resulting rotation?

b. Reasoning Use the results of part (a) and Problem 2. Make a conjecture about the center of rotation and the angle of rotation for any rotation that is the result of a composition of reflections across two intersecting lines.



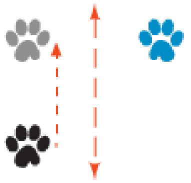
In a plane, any two congruent figures with the same orientation are related by either a translation or a rotation, and therefore by two reflections. Suppose two congruent plane figures A and B have opposite orientations. Reflect A and you get a figure A′ that has the same orientation as B. Thus, B is a translation or rotation image of A′. By Theorem 9-1, two reflections map A′ to B. The net result is that three reflections map A to B. This is summarized in what is sometimes called the Fundamental Theorem of Isometries.

take note

Theorem 9-3 Fundamental Theorem of Isometries

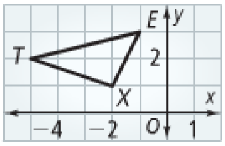
In a plane, one of two congruent figures can be mapped onto the other by a composition of at most three reflections.

If two figures are congruent and have opposite orientations (but are not simply reflections of each other), then there are a translation and a reflection that will map one onto the other. A **glide reflection** is the composition of a translation (a glide) and a reflection across a line parallel to the direction of translation. You can map a left paw print onto a right paw print with a glide reflection.



Problem 3 Finding a Glide Reflection Image

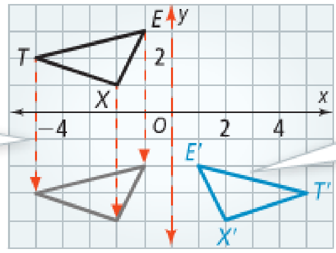
Coordinate Geometry **What is the image of $\triangle TEX$ for a glide reflection where the translation is $(x, y) \rightarrow (x, y - 5)$ and the line of reflection is $x = 0$?**



Know
- The vertices of $\triangle TEX$
- The translation rule
- The line of reflection

Need
The image of $\triangle TEX$ for the glide reflection

Plan
First use the translation rule to translate $\triangle TEX$. Then reflect the translation image of each vertex across the line of reflection.



Use the translation rule $(x, y) \rightarrow (x, y - 5)$ to move $\triangle TEX$ down 5 units.

Reflect the image of $\triangle TEX$ across the y-axis.

Got It? **3.** Graph $\triangle TEX$ from Problem 3. What is the image of $\triangle TEX$ for a glide reflection where the translation is $(x, y) \rightarrow (x + 1, y)$ and the line of reflection is $y = -2$?

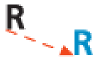
Essential Understanding You can map one of two congruent figures in a plane onto the other by a single reflection, translation, rotation, or glide reflection.

take note

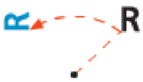
Theorem 9-4 Isometry Classification Theorem

There are only four isometries.

Translation

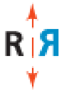


Rotation

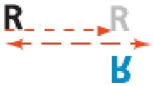


Orientations are the same.

Reflection



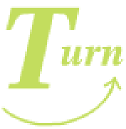
Glide Reflection



Orientations are opposite.

Problem 4 Classifying Isometries

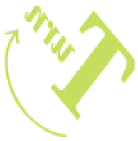
Each figure is an isometry image of the figure at the right. Are the orientations of the preimage and image the same or opposite? What type of isometry maps the preimage to the image?



Think

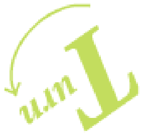
How do the arrows in the preimage and image help you?
Compare the directions of the arrows (counterclockwise or clockwise) to determine if the orientations of the figures are the same or different.

A



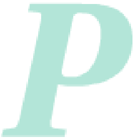
The orientations are opposite. The image is not a simple reflection of the preimage. The isometry that leads to opposite orientation but is not a reflection is a glide reflection.

B

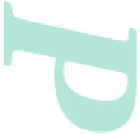


The orientations are the same. The image is not a translation of the preimage. The isometry that leads to the same orientation but is not a translation is a rotation.

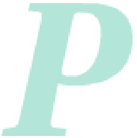
Got It? **4.** Each figure is an isometry image of the figure at the right. Are the orientations of the preimage and image the same or opposite? What type of isometry maps the preimage to the image?



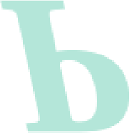
a.



b.



c.

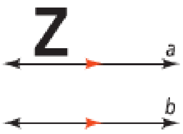


Lesson Check

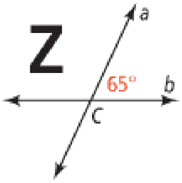
Do you know HOW?

Describe the image of Z reflected first across line *a* and then across line *b*.

1.



2.

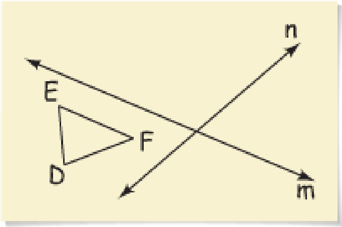


3. $\triangle PQR$ has vertices $P(0, 5)$, $Q(5, 3)$, and $R(3, 1)$. What are the vertices of the image of $\triangle PQR$ for a glide reflection where the translation is $(x, y) \rightarrow (x + 3, y - 1)$ and the reflection line is $y = -2$?

Do you UNDERSTAND?

4. Vocabulary In a glide reflection, what is the relationship between the direction of the translation and the line of reflection?

5. Error Analysis You reflect $\triangle DEF$ first across line m and then across line n. Your friend says you can get the same result by reflecting $\triangle DEF$ first across line n and then across line m. Explain your friend's error.



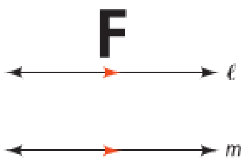
Practice and Problem-Solving Exercises

A Practice

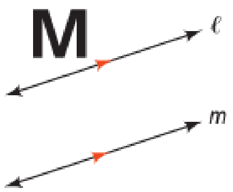
Find the image of each letter for a reflection first across line ℓ and then across line m. Is the resulting transformation a translation or a rotation? For a translation, describe the direction and distance. For a rotation, tell the center of rotation and the angle of rotation.

See Problems 1 and 2.

6.



7.



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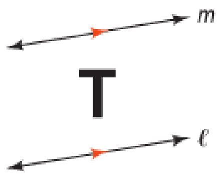
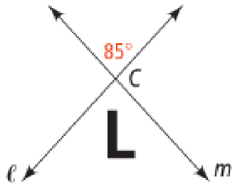
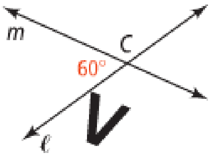


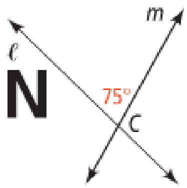
9.



10.



11.



Graph $\triangle PNB$ and its glide reflection image for the given translation and reflection line.

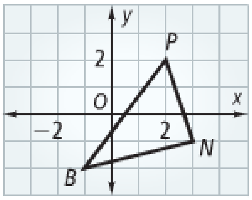
See Problem 3.

12. $(x, y) \rightarrow (x + 2, y)$; $y = 3$

13. $(x, y) \rightarrow (x, y - 3)$; $x = 0$

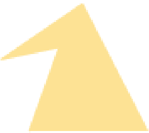
14. $(x, y) \rightarrow (x + 2, y + 2)$; $y = x$

15. $(x, y) \rightarrow (x - 1, y + 1)$; $y = -x$



Each figure below is an isometry image of the figure at the right. Tell whether the two figures' orientations are the same or opposite. Then classify the isometry.

See Problem 4.



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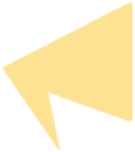

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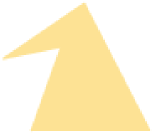

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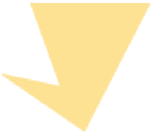

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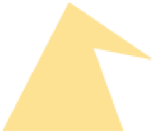
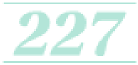


Each figure below is an isometry image of the figure at the right. Tell whether the two figures' orientations are the same or opposite. Then classify the isometry.



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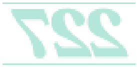

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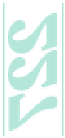

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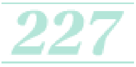
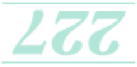


23.



B Apply

Use the given points and lines. Graph $\overline{AB}$ and its image $\overline{A'B'}$ after a reflection first across ℓ_1 and then across ℓ_2. Is the resulting transformation a translation or a rotation? For a translation, describe the direction and distance. For a rotation, tell the center of rotation and the angle of rotation.

24. $A(1, 5)$ and $B(2, 1)$; $\ell_1: x = 3$; $\ell_2: x = 7$

25. $A(2, 4)$ and $B(3, 1)$; ℓ_1: x-axis; ℓ_2: y-axis

26. $A(-4, -3)$ and $B(-4, 0)$; $\ell_1: y = x$; $\ell_2: y = -x$

27. $A(2, -5)$ and $B(-1, -3)$; $\ell_1: y = 0$; $\ell_2: y = 2$

28. $A(6, -4)$ and $B(5, 0)$; $\ell_1: x = 6$; $\ell_2: x = 4$

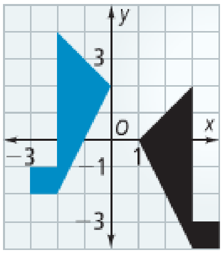
29. $A(-1, 0)$ and $B(0, -2)$; $\ell_1: y = -1$; $\ell_2: y = 1$

30. **Think About a Plan** $T \rightarrow T'(1, 5)$ by a glide reflection where the translation is $(x, y) \rightarrow (x + 3, y)$ and the line of reflection is $y = 1$. What are the coordinates of T?

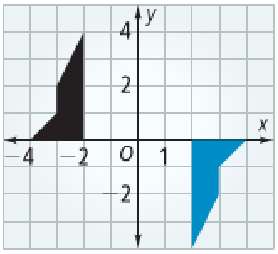
- How can you *work backwards* to find the coordinates of T?
- Should T be to left or to the right of T'?
- Should T be above or below T'?

The two figures are congruent. Describe the isometry that maps the black figure onto the blue figure.

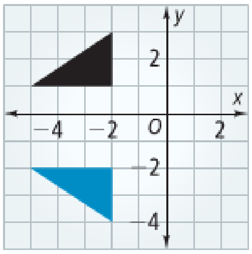
31.



32.



33. Which transformation maps the black triangle onto the blue triangle?



Ⓐ a glide reflection where the translation is $(x, y) \rightarrow (x, y - 3)$ and the line of reflection is $x = -2$

Ⓑ a 180° rotation about the origin

Ⓒ a reflection across the line $y = -\frac{1}{2}$

Ⓓ a reflection across the y-axis followed by a 180° rotation about the origin

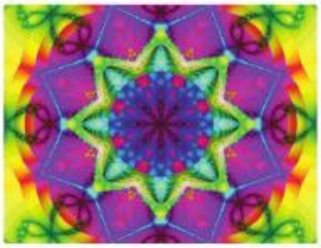
34. **Writing** Reflections and glide reflections are *odd isometries,* while translations and rotations are *even isometries.* Use what you have learned in this lesson to explain why these categories make sense.

35. **Open-Ended** Draw $\triangle ABC$. Describe a reflection, a translation, a rotation, and a glide reflection. Then draw the image of $\triangle ABC$ for each transformation.

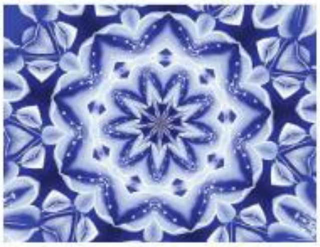
36. **Reasoning** The definition states that a glide reflection is the composition of a translation and a reflection. Explain why these can occur in either order.

Kaleidoscopes The vibrant images seen in a kaleidoscope are produced by compositions of reflections in intersecting mirrors. Determine the measure of the angle between the mirrors in each kaleidoscope image.

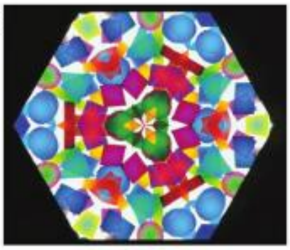
37.



38.



39.



Identify each mapping as a translation, reflection, rotation, or glide reflection. Find the translation rule, reflection line, center of rotation and angle of rotation, or glide translation rule and reflection line.

40. $\triangle ABC \rightarrow \triangle EDC$

41. $\triangle EDC \rightarrow \triangle PQM$

42. $\triangle MNJ \rightarrow \triangle EDC$

43. $\triangle HIF \rightarrow \triangle HGF$

44. $\triangle PQM \rightarrow \triangle JLM$

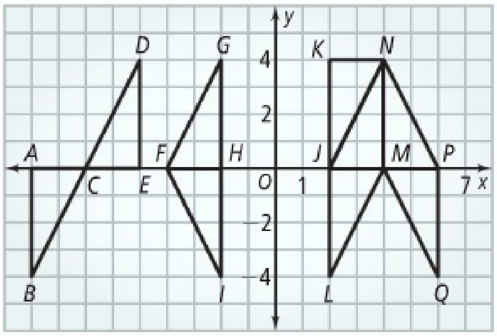
45. $\triangle MNP \rightarrow \triangle EDC$

46. $\triangle JLM \rightarrow \triangle MNJ$

47. $\triangle PQM \rightarrow \triangle KJN$

48. $\triangle KJN \rightarrow \triangle ABC$

49. $\triangle HGF \rightarrow \triangle KJN$



$P \rightarrow P'(3, -1)$ for the given translation and reflection line. Find the coordinates of P.

50. $(x, y) \rightarrow (x - 3, y); y = 2$

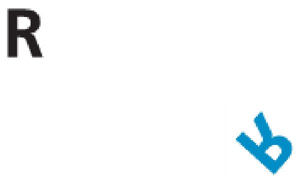
51. $(x, y) \rightarrow (x, y - 3); y = 2$

52. $(x, y) \rightarrow (x - 3, y - 3); y = x$

53. $(x, y) \rightarrow (x + 4, y - 4); y = -x$

Challenge

54. Describe a glide reflection that maps the black R to the blue R.



For the given transformation mapping $\overline{XY}$ to $\overline{X'Y'}$, prove that $\overline{XY} \cong \overline{X'Y'}$.

55. a translation

56. a reflection

57. a rotation

58. **Reasoning** Does an $x°$ rotation about a point P followed by a reflection across a line ℓ give the same image as a reflection across ℓ followed by an $x°$ rotation about P? Explain.

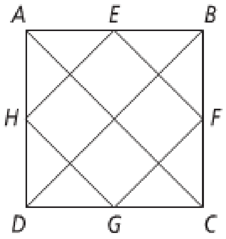
Standardized Test Prep

SAT/ACT

59. What is the image of $P(11, -5)$ for the translation $(x, y) \rightarrow (x - 12, y - 6)$, followed by a reflection across $x = 0$?

Ⓐ $(1, -11)$ Ⓑ $(-1, 11)$ Ⓒ $(1, 11)$ Ⓓ $(-1, -11)$

60. $ABCD$ is a rectangular window divided into 12 panes of glass. E, F, G, and H are midpoints of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{AD}$, respectively. Which statement must be true?



Ⓕ The quadrilateral panes are squares.

Ⓖ The quadrilateral panes are rhombuses.

Ⓗ The triangular panes are all congruent.

Ⓘ The triangular panes are right triangles.

61. A triangle has side lengths 7 in., 9 in., and x in. Which inequality must be true?

Ⓐ $7 < x < 9$ Ⓑ $-2 < x < 9$ Ⓒ $2 < x < 16$ Ⓓ $7 < x < 16$

62. $\triangle ABC$ has vertices $A(0, 5)$, $B(0, 0)$, and $C(5, 0)$. $\triangle PQR$ has vertices $P(0, 15)$ and $Q(0, 0)$. $\triangle ABC \sim \triangle PQR$. What are the coordinates of R?

Ⓕ $(0, -5)$ Ⓖ $(0, -15)$ Ⓗ $(15, -15)$ Ⓘ $(15, 0)$

Short Response

63. $\triangle ABC$ and $\triangle HIG$ are acute triangles such that $\triangle ABC \cong \triangle HIG$. $\overline{BL}$ and $\overline{IT}$ are altitudes of the two triangles. Is $\overline{BL} \cong \overline{IT}$? Justify your answer.

Mixed Review

Coordinate Geometry **Find the image of $\triangle ABC$ for a dilation with center $(0, 0)$ and the given scale factor.** **See Lesson 9-5.**

64. $A(0, 4)$, $B(0, 0)$, $C(-3, -1)$; scale factor 3

65. $A(4, 2)$, $B(2, 8)$, $C(8, 0)$; scale factor 1.5

66. $A(2, 3)$, $B(-4, -2)$, $C(5, -3)$; scale factor 2

67. $A(7, 8)$, $B(5, 4)$, $C(9, 6)$; scale factor 0.5

Identify the two statements that contradict each other. **See Lesson 5-5.**

68. I. $\triangle ABC$ is a right triangle.
II. $\triangle ABC$ is equiangular.
III. $\triangle ABC$ is isosceles.

69. I. In right $\triangle ABC$, $m\angle B = 90$.
II. In right $\triangle ABC$, $m\angle A = 80$.
III. In right $\triangle ABC$, $m\angle C = 90$.

Get Ready! To prepare for Lesson 9-7, do Exercises 70–78.

Classify the polygon with the given number of sides. **See Review, p. 57.**

70. five

71. eight

72. 12

Find the measure of an angle of each regular polygon. **See Lesson 6-1.**

73. triangle

74. quadrilateral

75. hexagon

76. octagon

77. decagon

78. 14-gon

Concept Byte

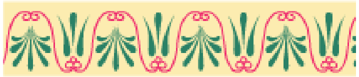
Use With Lesson 9-6

Frieze Patterns

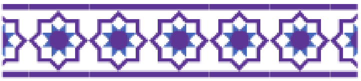
A frieze pattern is a design that repeats itself along a straight line. You can see frieze patterns in design trim. They often appear along the edges of buildings or as wallpaper borders of rooms. Every frieze pattern can be mapped onto itself by a translation.

Some frieze patterns can be mapped onto themselves by reflections.

A.

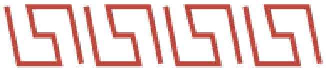


B.

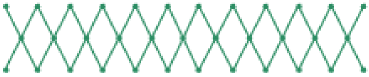


Some frieze patterns have repeated rotational symmetry.

C.

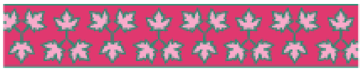


D.

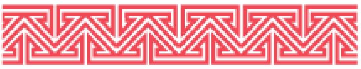


Some frieze patterns show glide reflections.

E.



F.



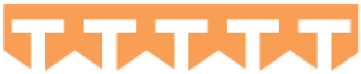
Exercises

For Exercises 1–4, refer to the frieze patterns above. You may find tracing paper helpful.

1. For each frieze pattern, find a portion (as small as possible) that you could translate repeatedly to form the entire pattern.
2. Which patterns show reflectional symmetry? Find their reflection lines.
3. Which patterns show rotational symmetry? Find their centers of rotation.
4. Which patterns show glide reflections? Find a glide translation and reflection line.

For each frieze pattern below, describe all transformations that map the pattern onto itself.

5.



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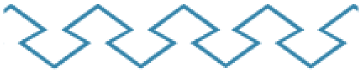

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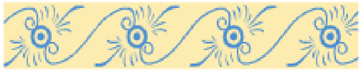

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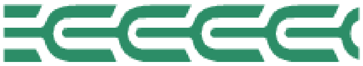
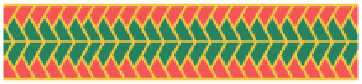


9.



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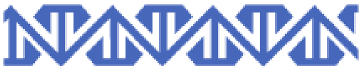


Concept Byte

Use With Lesson 9-7

ACTIVITY

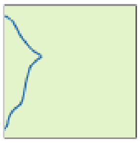
Creating Tessellations

The following activity shows the steps for making a creative *tessellation*, or repeating pattern of one or more shapes that fills a plane without leaving any gaps.

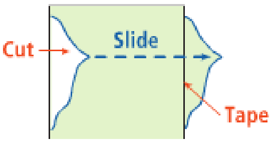
Activity

Step 1 Draw a 1.5-in. square on a blank piece of paper and cut it out.

Step 2 Draw a curve with endpoints on one side.

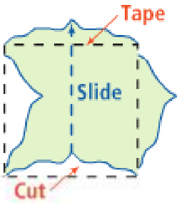


Step 3 Cut along the curve you drew and slide the cutout piece to the opposite side of the square. Tape it in place.

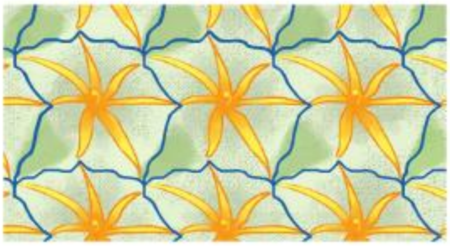


Step 4 Repeat this process using the other two opposite sides of the square.

Step 5 Rotate the resulting figure. What does the figure look like? Is it a penguin wearing a hat or a knight on horseback? Could it be a dog with floppy ears? Use your imagination and decorate your figure.



Step 6 Make a tessellation using your figure.



Exercises

Use the method described in the activity above to make a creative tessellation with each of the following types of polygons.

1. a rectangle with length 3 in. and width 2 in.
2. a parallelogram with 2-in. sides
3. a regular hexagon with 1.5-in. sides

9-7 Tessellations

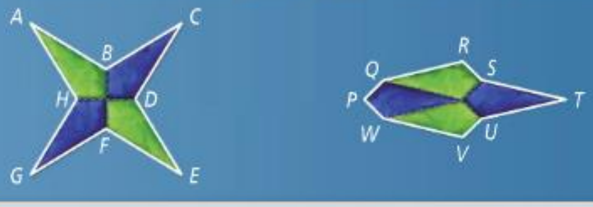
Objectives To identify figures that tessellate
To identify transformations and symmetries in tessellations

Quilts often use repeating patterns of shapes. There should be no gaps or overlaps.

SOLVE IT!

Getting Ready!

In the quilt block at the right, four congruent kites meet at one vertex, with white gaps around the vertex. In the two arrangements below, the kites meet at one vertex with no gaps. Consider each arrangement as a single shape (outlined in white). For which shape can you fit multiple copies together in a way that leaves no gaps? How do you know?



Lesson Vocabulary
- tessellation
- translational symmetry
- glide reflectional symmetry

A **tessellation,** or *tiling*, is a repeating pattern of figures that completely covers a plane, without gaps or overlaps. You can make tessellations with translations, rotations, and reflections. You can find shapes that *tessellate* in art (see Problem 1 below), nature (cells in a honeycomb), and everyday life (tiled floors).

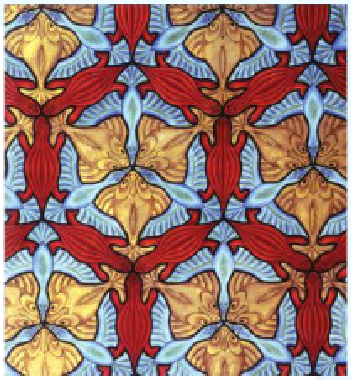
Essential Understanding Some shapes can fit together in a repeating pattern that fills a plane without gaps or overlaps. The angle measures of polygons that fit together in this way have a special relationship.

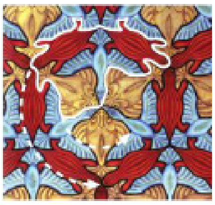
Think

How do you recognize the transformation? Comparing the positions of several pairs of corresponding points will help you see how the points moved.

Problem 1 Describing Tessellations

Art **What is the repeating figure in this tessellation? What transformation does the tessellation use?**

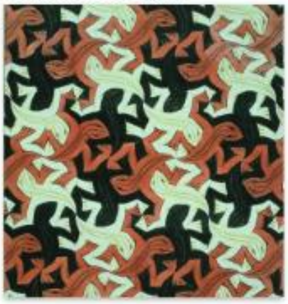




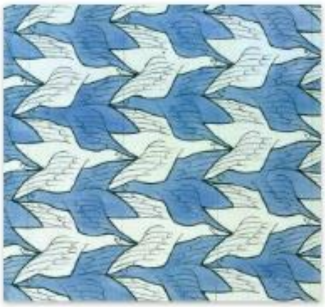
This tessellation uses a rotation. The fish, bird, and two turtles together form the repeating figure. The arrows show how each point of the repeating figure rotates.

Got It? **1.** What is the repeating figure in the tessellation? What transformation does the tessellation use?

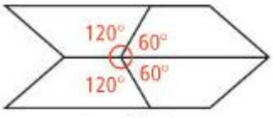
a.



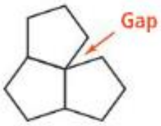
b.



The figures in a tessellation do not overlap or leave gaps. If the figures are polygons, the sum of the measures of the angles around any vertex in the tessellation must be 360.



Tessellation



Not a tessellation

Problem 2 Determining Whether a Figure Tessellates

Plan

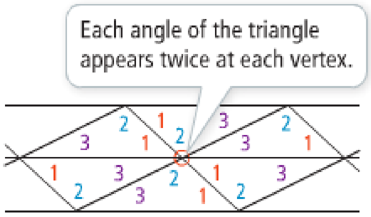
How can you use the fact that the sum of the measures of the angles around a vertex in a tessellation is 360? First, determine the measure of each angle of a regular 18-gon. Then check whether a multiple of this measure is 360.

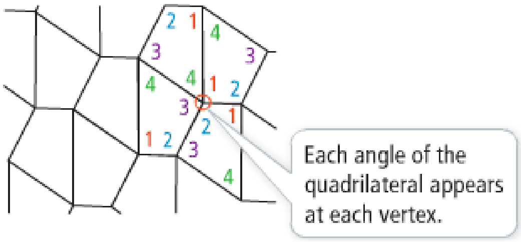
Does a regular 18-gon tessellate? Explain.

Think	Write
All angles of a regular 18-gon have the same measure.	Let a = the measure of one angle of a regular 18-gon.
Use the Polygon Angle-Sum Theorem to find *a*. Substitute 18 for *n* and simplify.	$a = \frac{180(n-2)}{n}$ $= \frac{180(18-2)}{18}$ $= 160$
Is there a multiple of 160 that equals 360?	$160 \cdot 2 = 320$ $160 \cdot 3 = 480$
Two copies of the 18-gon will leave gaps. Three copies of the 18-gon will overlap.	There is no multiple of 160 that equals 360, so a regular 18-gon does not tessellate.

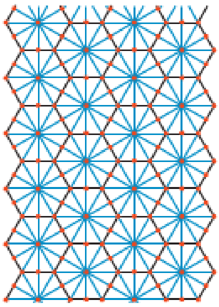
Got It? **2.** Does a regular hexagon tessellate? Explain.

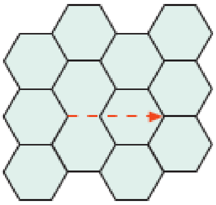
A figure does not have to be a regular polygon to tessellate. Every triangle and every quadrilateral tessellates, as the figures below show.

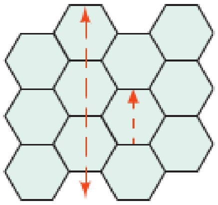

Each angle of the triangle appears twice at each vertex.


Each angle of the quadrilateral appears at each vertex.

Every tessellation has at least one type of symmetry. The tessellation with regular hexagons at the right has reflectional symmetry in each of the blue lines. It has rotational symmetry centered at each of the red points. The tessellation also has *translational symmetry* and *glide reflectional symmetry*, as shown below.

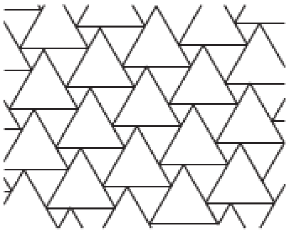




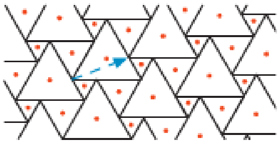


Problem 3 Identifying Symmetries in a Tessellation

What types of symmetry does the tessellation at the right have?



The tessellation has rotational symmetry, centered at each red point, and translational symmetry, as shown by the blue arrow.

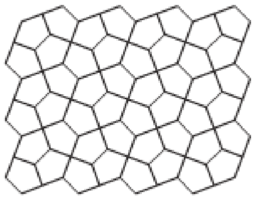


Think

How can you test the tessellation for the different types of symmetry?
Use tracing paper. Trace one unit of the repeating pattern and slide it into positions that allow you to check for the different types of symmetry.

Got It? 3. a. What types of symmetry does the tessellation at the right have?

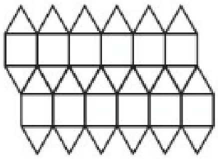
b. **Reasoning** The figure in Problem 3 shows one translation arrow. Copy the diagram and show translation arrows for two other translations that map the tessellation onto itself.



Lesson Check

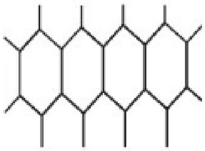
Do you know HOW?

1. What is the repeating figure in this tessellation? What transformation does the tessellation use?



2. Does a regular 15-gon tessellate? Explain.

3. What types of symmetry does the tessellation below have?



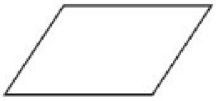
Do you UNDERSTAND?

4. Vocabulary A tessellation has translational symmetry. Must the tessellation also have glide reflectional symmetry? Explain.

5. Reasoning If you arrange three regular octagons so that they meet at one vertex, will they leave a gap or will they overlap? Explain.

6. Error Analysis A classmate says that a regular n-gon will tessellate if $3 \leq n \leq 6$. Explain your classmate's error.

7. Open-Ended Copy the parallelogram at the right and use it to make two different tessellations.



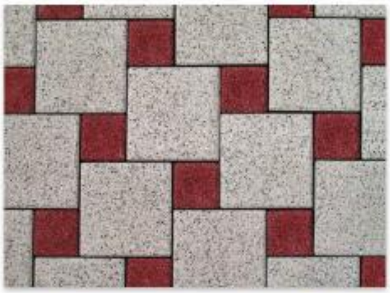
Practice and Problem-Solving Exercises

A Practice

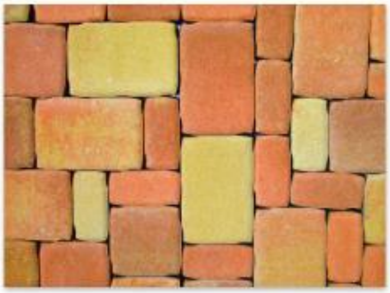
Does the picture show a tessellation of a repeating figure? If so, identify the repeating figure and the transformation used.

See Problem 1.

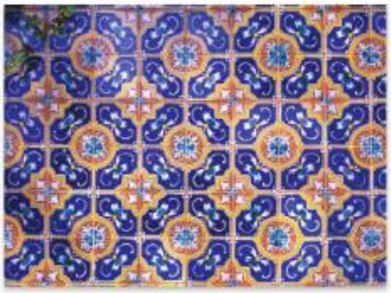
8.



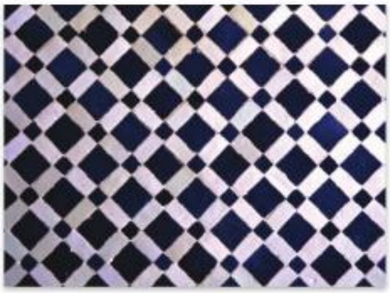
9.



10.



11.

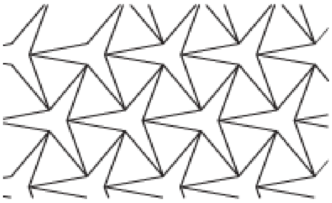


Determine whether each figure will tessellate a plane. Explain. **See Problem 2.**

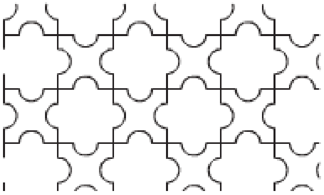
12. equilateral triangle
13. square
14. regular pentagon
15. regular heptagon
16. regular octagon
17. regular nonagon

List the types of symmetry each tessellation has. **See Problem 3.**

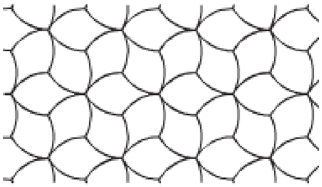
18.



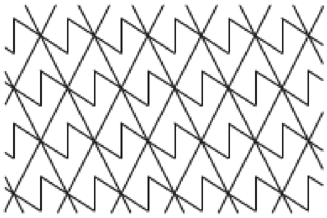
19.



20.



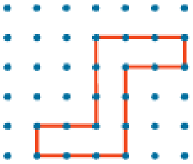
21.



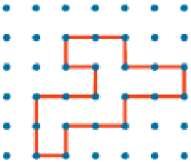
Apply

Use each figure to make a tessellation on dot paper.

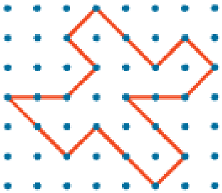
22.



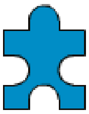
23.



24.



25. Which puzzle piece can tessellate a plane using *only* translation images of itself?

Ⓐ


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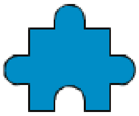

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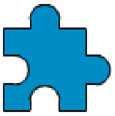

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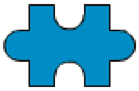
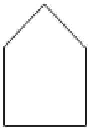


Show how to tessellate with each figure described below. Try to draw two different tessellations. If you think that two are not possible, explain.

26. a scalene triangle

27. the pentagon at the right

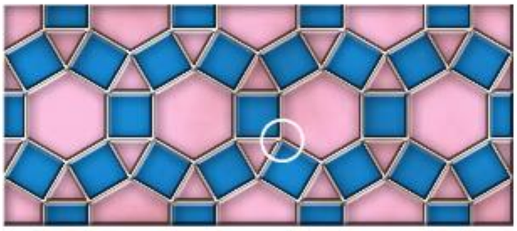


28. a quadrilateral with no sides parallel or congruent

29. **Writing** A *pure tessellation* is a tessellation made up of congruent copies of one figure. Explain why there are three, and only three, pure tessellations that use regular polygons. (*Hint*: See Exercises 12–17.)

30. **Think About a Plan** A *semiregular tessellation* is made up of two or more different regular polygons, with the same arrangement of the polygons at each vertex. Can you use regular octagons and squares, all with side length 1 unit, to make a semiregular tessellation? If so, draw a sketch.

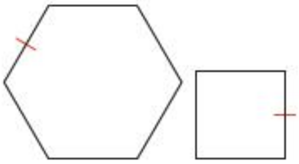
- Why is knowing that the figures have the same side length important?
- What is the sum of the measures of the angles at each vertex?



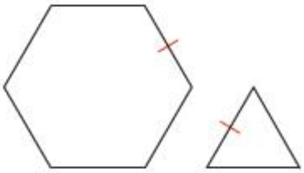
One hexagon, two squares, and one equilateral triangle meet at each vertex of this semiregular tessellation.

Can you make a semiregular tessellation using the given pair of regular polygons? If so, draw a sketch.

31.



32.



Determine whether each statement is *always, sometimes,* or *never* true. Explain.

33. A scalene triangle will tessellate.

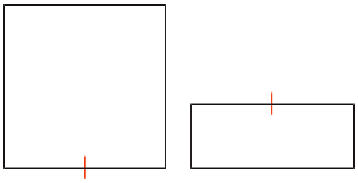
34. A rhombus will tessellate.

35. A hexagon will tessellate.

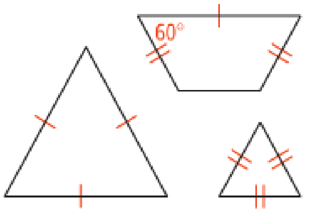
36. A regular decagon will tessellate.

Can each set of polygons be used to make a tessellation? If so, draw a sketch.

37.

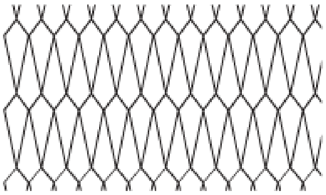


38.

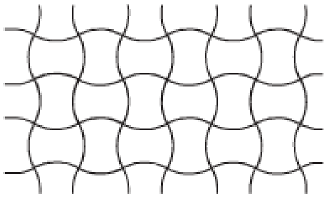


List the types of symmetry each tessellation has.

39.



40.



Challenge

41. Follow these steps to make a tessellation of a quadrilateral.

a. Draw quadrilateral $ABCD$ with no two sides congruent. Locate M, the midpoint of $\overline{AB}$, and N, the midpoint of $\overline{BC}$.

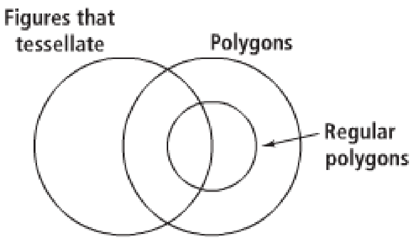
b. Draw the image of $ABCD$ for a 180° rotation about M.

c. Draw the image of $ABCD$ for a 180° rotation about N.

d. Draw the image of $ABCD$ for the translation that maps D to B.

42. List steps (like those in Exercise 41) that suggest a way to tessellate with any scalene triangle. Then list a second set of steps that suggest another way.

Copy the Venn diagram. Write each exercise number in the correct region of the diagram.



43. scalene triangle

44. obtuse triangle

45. kite

46. equilateral △

47. regular decagon

48. square

49. regular hexagon

50. regular octagon

51. rhombus

52. isosceles △

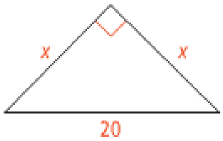
Standardized Test Prep

GRIDDED RESPONSE

SAT/ACT

53. What is the maximum number of sides less than 10 that a regular polygon can have and still tessellate a plane?

54. To the nearest hundredth, what is the value of x in the diagram at the right?



55. In $\triangle FGH$ and $\triangle XYZ$, $\angle G$ and $\angle Y$ are right angles. $\overline{FH} \cong \overline{XZ}$ and $\overline{GH} \cong \overline{YZ}$. If $GH = 7$ ft and $XY = 9$ ft, what is the area of $\triangle FGH$ in square inches?

56. $\triangle ACB$ is isosceles with base $\overline{AB}$. Point D is on $\overline{AB}$ and $\overline{CD}$ is the bisector of $\angle C$. If $CD = 5$ in. and $DB = 4$ in., what is BC to the nearest tenth of an inch?

57. Two angle measures of $\triangle JKL$ are 30 and 60. The shortest side measures 10 cm. What is the length, in centimeters, of the longest side of the triangle?

Mixed Review

58. A triangle has vertices $A(3, 2)$, $B(4, 1)$, and $C(4, 3)$. Find the coordinates of the images of A, B, and C for a glide reflection with translation $(x, y) \rightarrow (x, y + 1)$ and reflection line $x = 0$.

See Lesson 9-6.

The lengths of two sides of a triangle are given. What are the possible lengths for the third side?

See Lesson 5-6.

59. 16 in., 26 in.

60. 19.5 ft, 20.5 ft

61. 9 m, 9 m

62. $4\frac{1}{2}$ yd, 8 yd

Get Ready! **To prepare for Lesson 10-1, do Exercises 63–66.**

Find the area of each figure.

See Lesson 1-8.

63. a square with 5-cm sides

64. a rectangle with base 4 in. and height 7 in.

65. a 4.6 m-by-2.5 m rectangle

66. a rectangle with length 3 ft and width $\frac{1}{2}$ ft

9 *Pull It* All Together

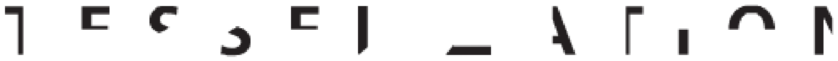
To solve these problems you will pull together many concepts and skills that you have studied about transformations.

BIG idea Visualization

You can use visualization to find the image of a figure for a transformation.

Task 1

Each figure below is part of a capital letter in the English alphabet. To find the whole letter, combine the figure with its image for the appropriate rotation or reflection. What letter corresponds to each figure? What transformation produces each letter?



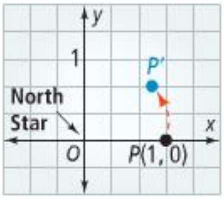
BIG idea Transformations

You can use transformations to describe a change in the position of a point.

Task 2

The arcs in the photo at the right appear to be paths of stars rotating about the North Star. To produce this effect, the photographer set a camera on a tripod and left the shutter open for an extended time. If the photographer left the shutter open for a full 24 hours, each arc would be a complete circle.

You can model a star's "rotation" in the coordinate plane. Place the North Star at the origin. Let $P(1, 0)$ be the position of the star at the moment the camera's shutter opens. Suppose the shutter is left open for 2 h 40 min, with the arc ending at P'.



a. What angle of rotation maps P onto P'?

b. What are the x- and y-coordinates of P' to the nearest thousandth?

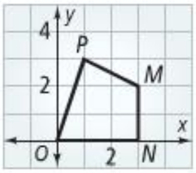
c. What translation rule maps P onto P'?

BIG idea Coordinate Geometry

You can use coordinate geometry to prove a dilation image is similar to its preimage.

Task 3

Copy the graph at the right. On the same set of axes, graph the image of $MNOP$ for a dilation with center $(0, 0)$ and scale factor 2. Use coordinate geometry and the definition of similar polygons to prove that $MNOP$ is similar to its image.



9 Chapter Review

Connecting BIG ideas and Answering the Essential Questions

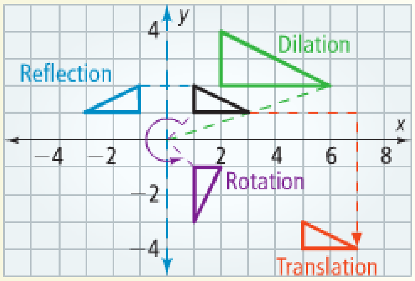
1 Transformations
When you translate, reflect, or rotate a geometric figure, its size and shape stay the same. When you dilate a geometric figure, the figure is enlarged or reduced.

2 Coordinate Geometry
You can show a transformation in the coordinate plane by graphing a figure and its image.

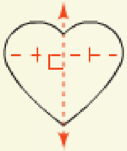
3 Visualization
Visualize a transformation that maps a figure onto itself.

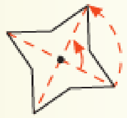
Transformations (Lessons 9-1, 9-2, 9-3, and 9-5)

The black triangle is the preimage of each transformation.



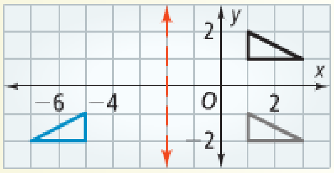
Symmetry (Lesson 9-4)


Line Symmetry


Rotational Symmetry

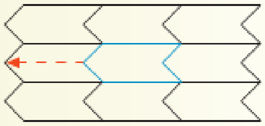
Composing Transformations (Lesson 9-6)

A glide reflection moves the black triangle down 3 units and then reflects it across the line $x = -2$.



Tessellations (Lesson 9-7)

A tessellation uses transformations of a figure to make a repeating pattern that fills a plane. Every tessellation has at least one type of symmetry.



Chapter Vocabulary

- center of a regular polygon (p. 560)
- composition of transformations (p. 547)
- dilation (p. 575)
- glide reflection (p. 587)
- glide reflectional symmetry (p. 597)
- image (p. 544)
- isometry (p. 544)
- line symmetry (p. 568)
- preimage (p. 544)
- reflection (p. 553)
- rotation (p. 559)
- rotational symmetry (p. 568)
- tessellation (p. 595)
- transformation (p. 544)
- translation (p. 546)
- translational symmetry (p. 597)

Choose the correct term to complete each sentence.

1. A(n) _?_ is a change in the position, shape, or size of a figure.

2. A(n) _?_ is a transformation in which the preimage and the image are congruent.

3. In a(n) _?_, all points of a figure move the same distance in the same direction.

4. A(n) _?_ is a translation followed by a reflection across a line parallel to the direction of the translation.

9-1 Translations

Quick Review

A **transformation** of a geometric figure is a change in its position, shape, or size. An **isometry** is a transformation in which the **preimage** and the **image** are congruent.

A **translation** is an isometry that maps all points of a figure the same distance in the same direction.

In a **composition of transformations,** each transformation is performed on the image of the preceding transformation.

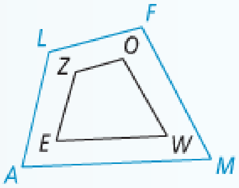
Example

What are the coordinates of the image of $A(5, -9)$ for the translation $(x, y) \rightarrow (x - 2, y + 3)$?

Substitute 5 for x and -9 for y in the rule.

$A(5, 9) \rightarrow (5 - 2, -9 + 3)$, or $A'(3, -6)$.

Exercises



5. **a.** A transformation maps $ZOWE$ onto $LFMA$. Does the transformation appear to be an isometry? Explain.

b. What is the image of $\overline{ZE}$? What is the preimage of M?

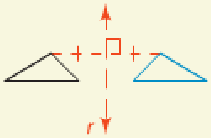
6. $\triangle RST$ has vertices $R(0, -4)$, $S(-2, -1)$, and $T(-6, 1)$. Graph the image of $\triangle RST$ for the translation $(x, y) \rightarrow (x - 4, y + 7)$.

7. Write a rule to describe a translation 5 units left and 10 units up.

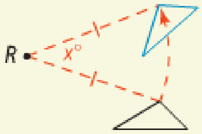
8. Find a single translation that has the same effect as the following composition of translations.
$(x, y) \rightarrow (x - 5, y + 7)$ followed by $(x, y) \rightarrow (x + 3, y)$

9-2 and 9-3 Reflections and Rotations

Quick Review



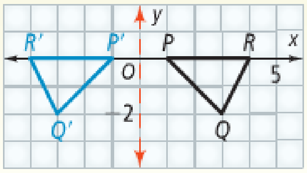
The diagram shows a **reflection** across line r. A reflection is an isometry in which a figure and its image have opposite orientations.



The diagram shows a **rotation** of $x°$ about point R. A rotation is an isometry in which a figure and its image have the same orientation.

Example

Use points $P(1, 0)$, $Q(3, -2)$, and $R(4, 0)$. What is the image of $\triangle PQR$ reflected across the y-axis?

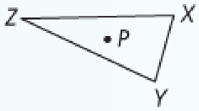


Graph $\triangle PQR$. Find P', Q', and R' such that the y-axis is the perpendicular bisector of $\overline{PP'}$, $\overline{QQ'}$, and $\overline{RR'}$. Draw $\triangle P'Q'R'$.

Exercises

Given points $A(6, 4)$, $B(-2, 1)$, and $C(5, 0)$, graph $\triangle ABC$ and its reflection image across each line.

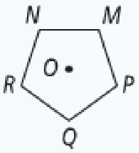
9. the x-axis **10.** $x = 4$ **11.** $y = x$



12. Copy the diagram. Then draw the image of $\triangle ZXY$ for a 90° rotation about P. Label the vertices of the image using prime notation.

13. Find the image of $P(-4, 1)$ for a 180° rotation about the origin.

Point O is the center of regular pentagon $NMPQR$.



14. What is the image of point N for a composition of a 72° rotation and a 144° rotation about O?

15. What is the angle of rotation that maps point P to point Q?

9-4 Symmetry

Quick Review

A figure has **reflectional symmetry** or **line symmetry** if there is a reflection for which it is its own image.

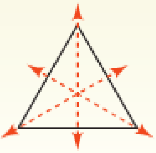
A figure that has **rotational symmetry** is its own image for some rotation of 180° or less.

A figure that has **point symmetry** has 180° rotational symmetry.

Example

How many lines of symmetry does an equilateral triangle have?

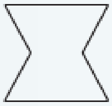
An equilateral triangle reflects onto itself across each of its three medians. The triangle has three lines of symmetry.



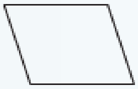
Exercises

Tell what type(s) of symmetry each figure has. If it has line symmetry, sketch the figure and the line(s) of symmetry. If it has rotational symmetry, state the angle of rotation.

16.



17.



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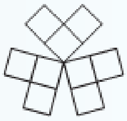


19. How many lines of symmetry does an isosceles trapezoid have?

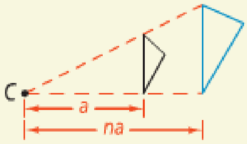
20. What type(s) of symmetry does a square have?

21. Give an example of a three-dimensional object that has rotational symmetry about a line.

9-5 Dilations

Quick Review

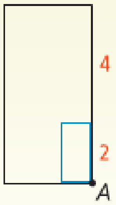
The diagram shows a **dilation** with center C and scale factor n. The preimage and image are similar.



In the coordinate plane, if the origin is the center of a dilation with scale factor n, then $P(x, y) \rightarrow P'(nx, ny)$.

Example

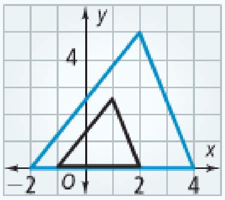
The blue figure is a dilation image of the black figure. The center of dilation is A. Is the dilation an enlargement or a reduction? What is the scale factor?



The image is smaller than the preimage, so the dilation is a reduction. The scale factor is $\frac{\text{image length}}{\text{original length}} = \frac{2}{2 + 4} = \frac{2}{6}$, or $\frac{1}{3}$.

Exercises

22. The blue figure is a dilation image of the black figure. The center of dilation is O. Tell whether the dilation is an enlargement or a reduction. Then find the scale factor.



Graph the polygon with the given vertices. Then graph its image for a dilation with center (0, 0) and the given scale factor.

23. $M(-3, 4)$, $A(-6, -1)$, $T(0, 0)$, $H(3, 2)$; scale factor 5

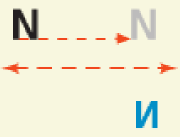
24. $F(-4, 0)$, $U(5, 0)$, $N(-2, -5)$; scale factor $\frac{1}{2}$

25. A dilation maps $\triangle LMN$ onto $\triangle L'M'N'$. $LM = 36$ ft, $LN = 26$ ft, $MN = 45$ ft, and $L'M' = 9$ ft. Find $L'N'$ and $M'N'$.

9-6 Compositions of Reflections

Quick Review

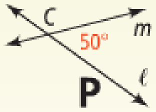
The diagram shows a **glide reflection** of N. A glide reflection is an isometry in which a figure and its image have opposite orientations.



There are exactly four isometries: translation, reflection, rotation, and glide reflection. Every isometry can be expressed as a composition of reflections.

Example

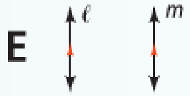
Describe the result of reflecting P first across line ℓ and then across line m.



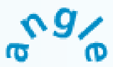
A composition of two reflections across intersecting lines is a rotation. The angle of rotation is twice the measure of the acute angle formed by the intersecting lines. P is rotated 100° about C.

Exercises

26. Sketch and describe the result of reflecting E first across line ℓ and then across line m.



Each figure is an isometry image of the figure at the right. Tell whether their orientations are the same or opposite. Then classify the isometry.



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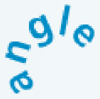

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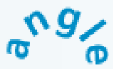

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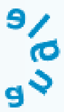


30. $\triangle TAM$ has vertices $T(0, 5)$, $A(4, 1)$, and $M(3, 6)$. Find the glide reflection image of $\triangle TAM$ for the translation $(x, y) \rightarrow (x - 4, y)$ followed by reflection across the line $y = -2$.

9-7 Tessellations

Quick Review

A **tessellation** is a repeating pattern of figures that completely covers a plane without gaps or overlaps. If the figures are polygons, the sum of the measures of the angles around any vertex in the tessellation is 360.

Every tessellation displays at least one type of symmetry: reflectional, rotational, translational, or glide reflectional symmetry.

Example

Does a regular decagon tessellate? Explain.

Use the Polygon Angle-Sum Theorem to find the measure, a, of each angle of a regular decagon.

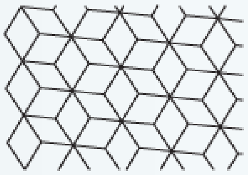
$$a = \frac{180(n - 2)}{n} = \frac{180(10 - 2)}{10} = 144$$

The sum of the angle measures around one vertex of a tessellation must be 360. 144 is not a factor of 360, so a regular decagon does not tessellate.

Exercises

For each tessellation, (a) identify the repeating figure and the transformation used, and (b) list the types of symmetry the tessellation has.

31.



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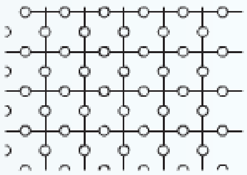
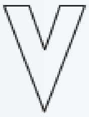


Determine whether each figure tessellates. If so, draw a sketch. If not, explain.

33. a kite

34. a regular 14-gon

35.



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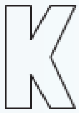

9 Chapter Test

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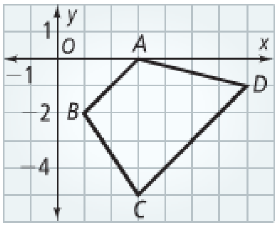
Do you know HOW?

For Exercises 1–7, find the coordinates of the vertices of the image of *ABCD* for each transformation.



1. reflection across the line $x = -4$

2. translation $(x, y) \rightarrow (x - 6, y + 8)$

3. rotation of 90° about the point (0, 0)

4. dilation with center (0, 0) and scale factor $\frac{2}{3}$

5. glide reflection with translation $(x, y) \rightarrow (x, y + 5)$ and reflection across the line $x = 0$

6. reflection across the line $y = x$

7. dilation with center (0, 0) and scale factor 3

8. Write the translation rule that maps $P(-4, 2)$ onto $P'(-1, -1)$.

What type of transformation has the same effect as each composition of transformations?

9. translation $(x, y) \rightarrow (x, y - 5)$ followed by reflection across the line $x = 6$

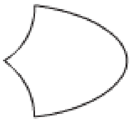
10. translation $(x, y) \rightarrow (x - 3, y + 2)$ followed by translation $(x, y) \rightarrow (x + 8, y - 4)$

11. reflection across the line $x = -2$ and then across the line $x = 4$

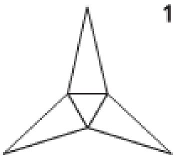
12. reflection across the line $y = -x$ and then across the line $y = x$

What type(s) of symmetry does each figure have?

13.



14.



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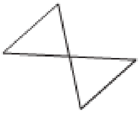
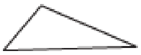


Determine whether each figure will tessellate. If so, draw a sketch. If not, explain.

16.



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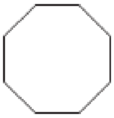

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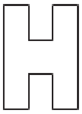
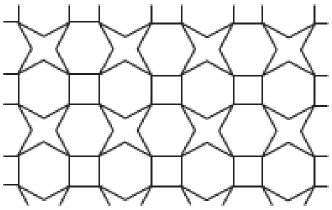
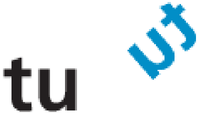


19. List the types of symmetry the tessellation has.



Identify the isometry that maps the black figure onto the blue figure.

20.



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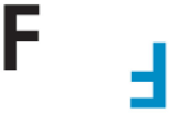


Do you UNDERSTAND?

22. Vocabulary Is a dilation an isometry? Explain.

23. Writing Line *m* intersects $\overline{UH}$ at *N*, and $UN = NH$. Must *H* be the reflection image of *U* across line *m*? Explain your reasoning.

24. Coordinate Geometry A dilation with center (0, 0) and scale factor 2.5 maps (a, b) onto $(10, -25)$. What are the values of *a* and *b*?

25. Error Analysis A classmate says that a certain figure has 50° rotational symmetry. Explain your classmate's error.

26. Reasoning Choose points *A*, *B*, and *C* in the first quadrant. Find the coordinates of A', B', and C' by multiplying the coordinates of *A*, *B*, and *C* by −2. What composition of transformations maps $\triangle ABC$ onto $\triangle A'B'C'$? Explain.

9 Cumulative Test Prep

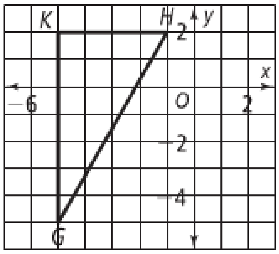
TIPS FOR SUCCESS

Some problems ask you to perform a transformation on a figure in the coordinate plane. Read the sample question at the right. Then follow the tips to answer it.

TIP 1

To calculate the scale factor of a dilation, you need to know the lengths of a pair of corresponding sides. You are given H'K'. You can use the graph to find HK.

$\triangle G'H'K'$ is the image of $\triangle GHK$ for a dilation with center $(0, 0)$ and $H'K' = 8$. What are the coordinates of H'?



Ⓐ $(-2, 4)$ Ⓒ $(1, 2)$

Ⓑ $(-1, -2)$ Ⓓ $(4, 2)$

TIP 2

Identify the coordinates of G, H, and K from the graph.

Think It Through

The scale factor is $\frac{H'K'}{HK} = \frac{8}{4} = 2$.

To find the coordinates of H', multiply the coordinates of H by the scale factor. H' is at $(2 \cdot (-1),\ 2 \cdot 2)$, or $(-2, 4)$. The correct answer is A.

Vocabulary Builder

As you solve problems, you must understand the meanings of mathematical terms. Match each term with its mathematical meaning.

A. centroid

B. circumcenter

C. line symmetry

D. point symmetry

E. proportion

F. tessellation

I. a reflection for which the figure is its own image

II. the point of concurrency of the medians in a triangle

III. an equation that states that two ratios are equal

IV. a repeating pattern of figures that completely covers a plane, without gaps or overlaps

V. a symmetry for which the figure is its own image for a 180° rotation

VI. the point of concurrency of the perpendicular bisectors of a triangle

Multiple Choice

Read each question. Then write the letter of the correct answer on your paper.

1. Which quadrilateral must have congruent diagonals?

Ⓐ kite Ⓒ parallelogram

Ⓑ rectangle Ⓓ rhombus

2. In a 30°-60°-90° triangle, the shortest leg measures 13 in. What is the measure of the longer leg?

Ⓕ 13 in. Ⓗ $13\sqrt{3}$ in.

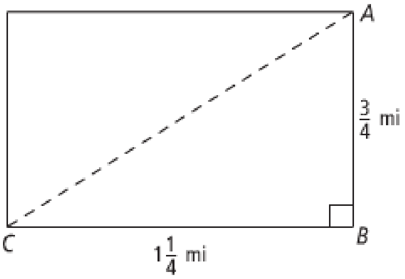
Ⓖ $13\sqrt{2}$ in. Ⓘ 26 in.

3. Three vertices of ▱$ABCD$ are $A(1, 7)$, $B(0, 0)$, and $C(7, -1)$. What are the coordinates of D?

Ⓐ $(6, 6)$ Ⓒ $(7, 7)$

Ⓑ $(6, 8)$ Ⓓ $(8, 6)$

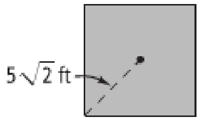
4. Mica and Joy are standing at corner A of the rectangular field shown below.



Mica walks diagonally across the field, from corner A to corner C. Joy walks from corner A to corner B, and then to corner C. To the nearest hundredth of a mile, how much farther did Joy walk than Mica did?

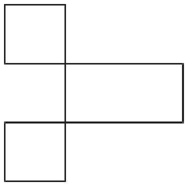
Ⓕ 0.54 mi
Ⓖ 1 mi
Ⓗ 1.46 mi
Ⓘ 2 mi

5. What is the area of the square?



Ⓐ 50 ft^2
Ⓑ 100 ft^2
Ⓒ $100\sqrt{2}$ ft^2
Ⓓ 150 ft^2

6. What type of symmetry does the figure have?



Ⓕ 60° rotational symmetry
Ⓖ 90° rotational symmetry
Ⓗ line symmetry
Ⓘ point symmetry

7. Which conditions allow you to conclude that a quadrilateral is a parallelogram?

Ⓐ one pair of sides congruent, the other pair of sides parallel
Ⓑ perpendicular, congruent diagonals
Ⓒ diagonals that bisect each other
Ⓓ one diagonal bisects opposite angles

8. Which statement is the contrapositive of the following conditional statement?

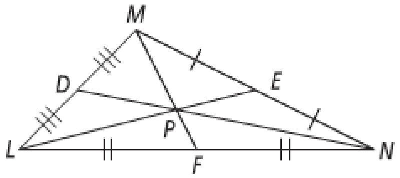
If Peg serves pudding, then it is not Laura's birthday.

Ⓕ If Peg serves pudding, then it is Laura's birthday.
Ⓖ If Peg does not serve pudding, then it is Laura's birthday.
Ⓗ If it is not Laura's birthday, then Peg serves pudding.
Ⓘ If it is Laura's birthday, then Peg does not serve pudding.

9. In a right triangle, which point lies on the hypotenuse?

Ⓐ incenter
Ⓑ orthocenter
Ⓒ centroid
Ⓓ circumcenter

10. In $\triangle LMN$, P is the centroid and $LE = 24$. What is PE?

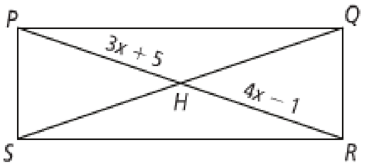


Ⓕ 8
Ⓖ 9
Ⓗ 10
Ⓘ 16

11. What is the sum of the angle measures of a 32-gon?

Ⓐ 3200
Ⓑ 3800
Ⓒ 5400
Ⓓ 5580

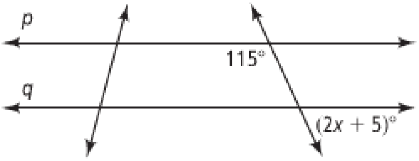
12. The diagonals of rectangle $PQRS$ intersect at H. What is the length of $\overline{QS}$?



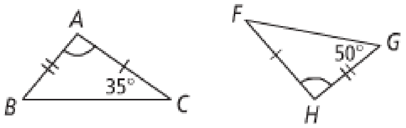
Ⓕ 6
Ⓖ 12
Ⓗ 23
Ⓘ 46

GRIDDED RESPONSE

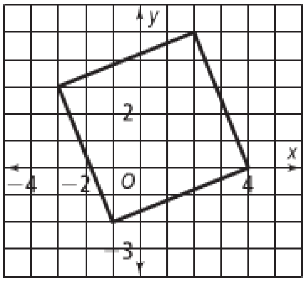
13. What is the value of x for which $p \parallel q$?



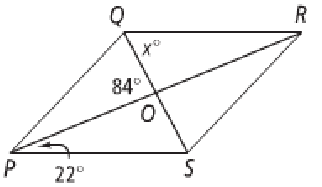
14. What is the measure of $\angle H$?



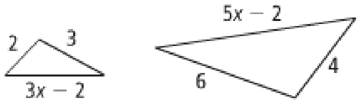
15. What is the area of the square, in square units, in the figure below?



16. In $\square PQRS$, what is the value of x?



17. For what value of x are the two triangles similar?



18. Your friend is 5 ft 6 in. tall. When your friend's shadow is 6 ft long, the shadow of a nearby sculpture is 30 ft long. What is the height, in feet, of the sculpture?

19. Lucia makes a triangular garden in one corner of her fenced rectangular backyard. She has 25 ft of edging to use along the unfenced side of the garden. One of the fenced sides of the garden is 15 ft long. What is the length, in feet, of the other fenced side of the garden?

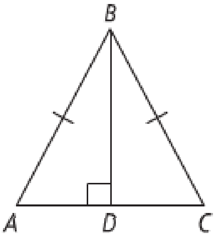
Short Response

20. $\triangle DEB$ has vertices $D(3, 7)$, $E(1, 4)$, and $B(-1, 5)$. In which quadrant(s) is the image of $\triangle DEB$ for a 270° rotation about the origin? Draw a diagram.

21. What is the area of an isosceles right triangle whose hypotenuse is $7\sqrt{2}$? Show your work.

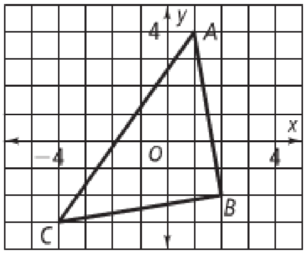
22. The coordinates of the endpoints of $\overline{CD}$ are $C(5, 2.5)$ and $D(0, -9.5)$. Find the length of $\overline{CD}$ and the coordinates of the midpoint of $\overline{CD}$. Show your work.

23. In $\triangle ABC$ below, $\overline{AB} \cong \overline{CB}$ and $\overline{BD} \perp \overline{AC}$. Prove that $\triangle ABD \cong \triangle CBD$.



Extended Response

24. Is $\triangle ABC$ a right triangle? Justify your answer.



25. The capital letter T shown at the right is made up of five congruent squares. Sketch a tessellation using congruent copies of this T. What symmetries does your tessellation have?